An infinite system of Brownian balls with infinite range interaction

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Abstract

We study an infinite system of Brownian hard balls, moving in $\mathbb{R}^d$ and submitted to a smooth infinite range pair potential. It is represented by a diffusion process, which is constructed as the unique strong solution of an infinite dimensional Skorohod equation. We also prove that canonical Gibbs states associated to the sum of the hard core potential and the pair potential are reversible measures for the dynamics.

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0 Introduction

We consider a system of infinitely many indistingable hard balls with diameter $r > 0$ in a d-dimensional Euclidean space $\mathbb{R}^d$, $d \geqslant 2$, undergoing Brownian motions and submitted to the influence of a smooth infinite range pair potential Φ_s.

Infinite systems of interacting Brownian particles (i.e. balls with diameter reduced to 0) have been treated by Lang [Lan77a, Lan77b] and Fritz [Fri87] in the case of a smooth nonnegative pair potential with finite range. Tanemura [Tan96] studied the case of Brownian hard balls without supplementary pair potential. Recently, Fradon and Roelly [FR] analyzed an infinite system of hard balls submitted to a smooth finite range pair potential under the assumption that the density of balls is sufficiently small. Here, we present a generalization of the previous works with respect to two important points : the spatial mean density of the initial configuration is arbitrary large (when it is Gibbsian, this means that there is no restriction on the activity), and the balls interact even if they are separated by any large distance (the potential Φ_s has infinite range with exponential decrease). Under these assumptions, we construct the gradient diffusion $\mathbf{X}(t) = (X_i(t), i \in \mathbb{N}, t > 0)$, unique strong solution of the following infinite-dimensional Skorohod type system of equations :

$$X_i(t) = X_i + B_i(t) - \frac{1}{2} \sum_{j \in \mathbb{N}} \int_0^t \nabla \Phi_s(X_i(s) - X_j(s)) ds + \sum_{j \in \mathbb{N}} \int_0^t (X_i(s) - X_j(s)) dL_{ij}(s),$$

where, for any $i, j \in \mathbb{N}, i \neq j, t \geqslant 0, \quad |X_i(t) - X_j(t)| \geqslant r$, and $L_{ij}(t)$ are local times, that is nondecreasing continuous processes with

$$L_{ij}(0) = 0, L_{ij}(.) = L_{ji}(.) \text{ and } L_{ij}(t) = \int_0^t 1\!\!1_{\{r\}}(|X_i(s) - X_j(s)|) dL_{ij}(s).$$

In a first section we define state spaces and present the main results of the paper. In the second section the finite dimensional Skohorod problem is stated, some geometrical aspect of the configuration space is discussed, and the dynamics of finitely many hard balls is solved. The third section is devoted to the convergence of finite-dimensional approximations towards $\mathbf{X}(.)$, a reversible solution of the above equation. In the last section, we prove some measurability properties of the diffusion $\mathbf{X}(.)$ and the existence of solutions with deterministic initial conditions.

1 Statement of the results

1.1 Configuration spaces and path spaces

In the whole paper, $|.|$ denotes the euclidean norm and $\langle \cdot, \cdot \rangle$ denotes the corresponding scalar product.
Let $\mathfrak{M}$ be the set of all countable subsets $\eta = \{\eta_i\}_i$ of $\mathbb{R}^d$ satisfying $N_\Lambda(\eta) \equiv \sharp(\eta \cap \Lambda) < +\infty$ for any compact set Λ of $\mathbb{R}^d$. We equivalently consider $\eta \in \mathfrak{M}$ as a non-negative integer valued Radon measure on $\mathbb{R}^d$: $\eta = \sum_i \delta_{\eta_i}$. $\mathfrak{M}$ is endowed with the topology of vague convergence.

The particles we deal with in the present paper are hard balls of radius $r/2$ (for a fixed $r > 0$) evolving in $\mathbb{R}^d$. So the configuration space of the system is the following (compact) subset of $\mathfrak{M}$:

$$\mathfrak{X} = \{\eta = \{\eta_i\}_{i\in \mathbf{J}} \in \mathfrak{M} \text{ where } \mathbf{J} \subset \mathbb{N} \text{ and for } i \neq j, |\eta_i - \eta_j| \geqslant r\},$$

where η_i are the positions of the centers of the hard balls.

Throughout this paper when S is a topological space, we denote by $\mathcal{B}(S)$ the topological Borel field of S, and by $W(S)$ the set of all S-valued continuous functions defined on $[0, \infty)$. $W(S)$ is endowed with the local uniform topology.

The σ-field $\sigma(N_A; A \in \mathcal{B}(\mathbb{R}^d))$ coincides with $\mathcal{B}(\mathfrak{X})$. We will also use the σ-field $\mathcal{B}_\Lambda(\mathfrak{X})$ defined for each compact subset Λ of $\mathbb{R}^d$ by

$$\mathcal{B}_\Lambda(\mathfrak{X}) = \sigma(N_A; A \in \mathcal{B}(\mathbb{R}^d), A \subset \Lambda).$$

We introduce the following measurable subsets of $W(\mathfrak{X})$.
For $\varepsilon > 0$, $0\leqslant s < t < \infty$ and a bounded open subset O of $\mathbb{R}^d$, we denote by $\mathcal{C}(\varepsilon, [s,t], O)$ the set of all paths $\mathbf{x}(\cdot)$ of $W(\mathfrak{X})$ such that on the time interval $[s,t]$ the balls stay at distance greater than $\varepsilon/2$ from the boundary of O :

$$\begin{array}{ll} \text{if } j \in \mathbf{J}(\mathbf{x}(s), O), & \forall u \in [s,t],\ U_{\frac{r+\varepsilon}{2}}(x_j(u)) \subset O \\ \text{if } j \notin \mathbf{J}(\mathbf{x}(s), O), & \forall u \in [s,t],\ U_{\frac{r+\varepsilon}{2}}(x_j(u)) \subset \mathbb{R}^d \setminus O. \end{array}$$

Here

$$\mathbf{J}(\eta = \{\eta_i\}_i, O) = \{i \in \mathbb{N} : \eta_i \in O\}$$

and, for $\alpha > 0, U_\alpha(A)$ denotes the open α-neighbourhood of a set $A \subset \mathbb{R}^d$. $U_\alpha(x)$ is the abbreviated form of $U_\alpha(\{x\})$, and, for simplicity, we just write U_α instead of $U_\alpha(\{0\})$. So,

$$U_\alpha = \{x \in \mathbb{R}^d, |x| < \alpha\}.$$

For $\varepsilon, \delta > 0$, $T, M \in \mathbb{N}$ and $\ell \in \mathbb{N}$, we denote by $\mathcal{C}[\varepsilon, \delta, T, M, \ell]$ the set of all paths $\mathbf{x}(\cdot)$ of $W(\mathfrak{X})$ such that for any $k = 0, 1, \ldots, [\frac{T}{\delta}]$, there exists a sequence $O_k^1, O_k^2, \ldots O_k^Q$ of bounded open disjoint subsets of $\mathbb{R}^d$ verifying

$$\begin{array}{lll} (1.1) & \forall q \in \{1,2,\ldots,Q\} & \mathbf{x}(\cdot) \in \mathcal{C}(\varepsilon, [k\delta, (k+1)\delta], O_k^q), \\ (1.2) & \bigcup_{q=1}^Q \mathbf{J}(\mathbf{x}(k\delta), O_k^q) & \supset\ \mathbf{J}(\mathbf{x}(k\delta), U_\ell), \\ (1.3) & \forall q \in \{1,2,\ldots,Q\} & 1 \leqslant \sharp \mathbf{J}(\mathbf{x}(k\delta), O_k^q) \leqslant M. \end{array}$$

We now define a measurable subset $\mathcal{C}$ of $W(\mathfrak{X})$ which will be a path space containing the processes studied in this paper :

$$\mathcal{C} = \bigcup_{\kappa\in(0,\frac{1}{2})} \bigcap_{p=1}^{\infty} \bigcup_{M=1}^{\infty} \bigcap_{T=1}^{\infty} \bigcap_{m_0=1}^{\infty} \bigcup_{m=m_0}^{\infty} \mathcal{C}(m^{-\kappa}, \frac{1}{m}, T, M, m^p). \tag{1.4}$$

1.2 Description of the potential and associated Gibbs states

We are dealing with a pair potential $\Phi = \Phi_h + \Phi_s$, where Φ_h represents a hard core repulsion, i.e.

$$(1.5) \qquad \Phi_h(\xi_i, \xi_j) = \begin{cases} 0 & \text{if } |\xi_i - \xi_j| \geqslant r, \\ +\infty & \text{otherwise,} \end{cases}$$

and $\Phi_s(\xi_i, \xi_j) = \Phi_s(\xi_i - \xi_j)$ is an $\mathbb{R}$-valued $\mathcal{C}^1$-pair potential on $\mathbb{R}^d$ satisfying the following assumptions (1.6), (1.7) and (1.8):

- Summability on $\mathfrak{X}$ of the functions Φ_s and $\nabla\Phi_s$:

$$(1.6) \qquad \forall \{\xi_j\}_j \in \mathfrak{X},\ \forall i,\ \sum_{j \neq i} |\Phi_s(\xi_i - \xi_j)| < +\infty \text{ and } \sum_{j \neq i} |\nabla\Phi_s(\xi_i - \xi_j)| < +\infty$$

- Lipschitzianity of $\nabla\Phi_s$ on finite allowed configurations :
  There exists K such that for each finite subset $\mathbf{J}$ of $\mathbb{N}$, each $\xi, \eta \in \mathfrak{X}$ and each $i \in \mathbb{N}$ verifying $\max\limits_{j \in \mathbf{J},\ j \neq i} |(\xi_i - \xi_j) - (\eta_i - \eta_j)| < r/2$, one has :

$$(1.7) \qquad \sum_{j \in \mathbf{J},\ j \neq i} |\nabla\Phi_s(\xi_i - \xi_j) - \nabla\Phi_s(\eta_i - \eta_j)| \leqslant K \max_{j \in \mathbf{J},\ j \neq i} |(\xi_i - \xi_j) - (\eta_i - \eta_j)|$$

- Stretched exponential decrease on $\mathfrak{X}$ of the sum of $\nabla\Phi_s$:

$$(1.8) \qquad \begin{array}{l} \exists \beta_0, \beta_1, \beta_2 > 0, \text{ such that for } R \text{ large enough, } \forall \xi \in \mathfrak{X}, \forall i \\ \displaystyle\sum_{j, |\xi_i - \xi_j| > R} |\nabla\Phi_s(\xi_i - \xi_j)| \leq g(R) \equiv \beta_0 \exp(-\beta_1 R^{\beta_2}) \end{array}$$

The range of the smooth pair potential Φ_s may be finite or infinite, i.e. the support of Φ_s may be compact or not. If the range of Φ_s is finite, the function g appearing in (1.8) vanishes.

Remark 1.1 *By an elementary comparison argument with the summability on the lattice $r\mathbb{Z}^d$, inequalities (1.6) on Φ_s and $\nabla\Phi_s$ are equivalent to the following uniform summability on $\mathfrak{X}$:*

$$(1.9) \qquad \overline{\Phi_s} = \sup_{\xi \in \mathfrak{X}} \sup_{i \in \mathbb{N}} \sum_{j \neq i} |\Phi_s(\xi_i - \xi_j)| < +\infty \ \textit{and}\ \overline{\nabla\Phi_s} = \sup_{\xi \in \mathfrak{X}} \sup_{i \in \mathbb{N}} \sum_{j \neq i} |\nabla\Phi_s(\xi_i - \xi_j)| < +\infty$$

A sufficient condition for (1.6) and (1.7) to hold is that Φ_s has $\mathcal{C}^2$-regularity and :

$$(1.10) \qquad \sum_{k \in \mathbb{Z}^d} |\Phi_s(rk)| < +\infty, \sum_{k \in \mathbb{Z}^d} |\nabla\Phi_s(rk)| < +\infty \ \textit{and} \ \sum_{k \in \mathbb{Z}^d} \sup_{x \in [0,r]^d} |D^2\Phi_s(rk + x)| < +\infty.$$

Remark 1.2 *Let us suppose that the infinite range pair potential $\Phi_s(x), x \in \mathbb{R}^d$, is a function of the norm of x, in such a way that there exists a function φ on $\mathbb{R}^+$ verifying $\Phi_s(x) = \varphi(|x|)$. Such assumption is physically very natural. Then, the following regularity of the function φ is sufficient to imply (1.6) and (1.7) :*
φ is $\mathcal{C}^2, |\varphi|, |\varphi'|$ and $|\varphi''|$ are non increasing functions on some interval $[R, +\infty[$ and

$$\int_{[R,+\infty[} |\varphi(u)| u^{d-1} du < +\infty.$$

To obtain the exponential bound (1.8), it is enough to suppose the following exponential decreasing property of φ' :

$$\exists R, \gamma_0, \gamma_1, \gamma_2 > 0, \forall u > R, |\varphi'(u)| \leq \gamma_0 \exp(-\gamma_1 u^{\gamma_2}).$$

Now, we can define the set of Gibbs states associated to the pair potential $\Phi = \Phi_h + \Phi_s$. For $\xi = \{\xi_1, \xi_2, \dots, \xi_n\}$ and $\eta \in \mathfrak{M}$ we define

$$\chi(\xi|\eta) = \exp\{- \sum_{1 \leqslant i < j \leqslant n} \Phi(\xi_i, \xi_j) - \sum_{i=1}^{n} \sum_{j} \Phi(\xi_i, \eta_j)\}.$$

By assumption (1.6) the second (infinite) summation on η_j is bounded and then, the function χ is well defined.
Let us fix a real positive number z. For any compact subset $\Lambda \subset \mathbb{R}^d$, we denote by $\lambda_{\Lambda,z}$ the Poisson distribution on $\mathfrak{M}(\Lambda)$ with intensity measure zdx on Λ, where dx denotes the Lebesgue measure, and $\mathfrak{M}(\Lambda)$ is the set of all finite subsets of Λ.

Definition 1.3 *A probability measure μ on $\mathfrak{X}$ is called a Gibbs state with respect to the activity $z \geqslant 0$ and the potential Φ, if μ satisfies the following DLR equation for any compact subset Λ of $\mathbb{R}^d$:*

$$\mu(\cdot|\mathcal{B}_{\Lambda^c}(\mathfrak{X}))(\eta) = \mu_{\Lambda,\eta,z}(\cdot), \qquad \eta \quad \mu\text{-a.s.},$$

where $\mu_{\Lambda,\eta,z}$ is the probability measure on $\mathfrak{M}(\Lambda)$ defined by

$$(1.11) \qquad \mu_{\Lambda,\eta,z}(d\xi) = \frac{1}{Z_{\Lambda,\eta,z}} \chi(\xi|\eta \cap \Lambda^c) \lambda_{\Lambda,z}(d\xi),$$

$$\textit{and} \qquad Z_{\Lambda,\eta,z} = \int_{\mathfrak{M}(\Lambda)} \chi(\xi|\eta \cap \Lambda^c) \lambda_{\Lambda,z}(d\xi).$$

The set of such Gibbs states is denoted by $\mathcal{G}(z, \Phi)$.

The set $\mathcal{G}(z, \Phi)$ is convex and compact with respect to the topology of weak convergence. Since Φ_s, the smooth part of the potential, satisfies the condition (1.6) and then the condition (1.9), it is a stable potential in the sense of Ruelle with stability constant $\overline{\Phi_s}$ and then Φ is superstable (See [Rue69] §3.2.5). This assures the existence of at least one element in $\mathcal{G}(z, \Phi)$, i.e.

$$\mathcal{G}(z, \Phi) \neq \emptyset.$$

About the cardinality of $\mathcal{G}(z,\Phi)$, we do the following remarks:

- if z is smaller than a critical value z_c, Ruelle proved that uniqueness holds (See [Rue69] Theorem 4.2.3). Moreover, he did explicit a lowerbound for z_c : in our case,

$$z_c \geqslant \Big(\exp(2\overline{\Phi_s}+1) \int_{\mathbb{R}^d} |1-\exp -\Phi(x)| dx \Big)^{-1}.$$

- for z large enough, it is a well known conjecture that the set of extremal points of $\mathcal{G}(z,\Phi)$ has a cardinal greater than 2 (See Ruelle [Rue69], Georgii [Geo88]), in other words a phase transition should occur.

1.3 The infinite dimensional diffusion

Let $(\Omega,\mathcal{F},P)$ be a probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ such that each $\mathcal{F}_t$ contains all P-negligible sets.

Let $(B_i(t), i\in\mathbb{N}, t\geqslant 0)$ be a sequence of independent d-dimensional $\mathcal{F}_t-$Brownian motions and $\mathbf{X}_0=\{X_1,X_2,\dots\}$ be an $\mathcal{F}_0$-measurable $\mathfrak{X}$-valued random variable on $(\Omega,\mathcal{F},P)$.

We consider the following system of equations : for each $i\in\mathbb{N}, t>0$,

$$X_i(t) = X_i + B_i(t) - \frac{1}{2}\sum_{j\in\mathbb{N}}\int_0^t \nabla\Phi_s(X_i(s)-X_j(s))ds + \sum_{j\in\mathbb{N}}\int_0^t (X_i(s)-X_j(s))dL_{ij}(s) \tag{1.12}$$

where

$$(\mathbf{X}(t), t\geqslant 0) \text{ is an } \mathfrak{X}-\text{valued process and} \tag{1.13}$$

$$L_{ij}(t), i,j\in\mathbb{N}, \text{ are nondecreasing continuous processes with} \tag{1.14}$$

$$L_{ij}(0)=0, L_{ij}(.)=L_{ji}(.) \text{ and } L_{ij}(t)=\int_0^t \mathbb{I}_{\{r\}}(|X_i(s)-X_j(s)|)dL_{ij}(s).$$

The pair $(\mathbf{X}(.),\mathbf{L}(.))=\{(X_i(.),L_{ij}(.)), i,j\in\mathbb{N}\}-$ or simply $\mathbf{X}(.)-$ is called a solution of (1.12) provided that (1.13) and (1.14) are satisfied and that $\mathbf{X}(\omega,\cdot)\in\mathcal{C}$ for P-almost all ω, where $\mathcal{C}$ is the set of regular paths defined by (1.4).

We are now ready to state the main results of this paper.

Theorem 1.4 (i) *There exists a measurable subset $\mathfrak{Y}$ of $\mathfrak{X}$ such that for each fixed initial value $\mathbf{X}_0$ in $\mathfrak{Y}$ equation* (1.12) *admits a unique solution $\mathbf{X}(.)$ which is an $\mathfrak{Y}$-valued diffusion process.*

(ii) *If the law of the initial variable $\mathbf{X}_0$ is a Gibbs state of $\mathcal{G}(z,\Phi)$ for some $z\in(0,\infty)$, then $P(\mathbf{X}_0\in\mathfrak{Y})=1$ and the $\mathfrak{X}$-valued process $\mathbf{X}(.)$ is a reversible diffusion process.*

So, completing the previous remark on the non-uniqueness of Gibbs states with large activity, we deduce that for large z there may exist several reversible diffusion processes solutions of equation (1.12).

Let us mention that using Dirichlet forms, one can construct the law of such a diffusion (cf [Osa96], [Yos96]). In [Tan97], one of the authors (H.T.) used a Skorohod type decomposition to prove that, in the case $\Phi_s \equiv 0$, the diffusion associated with the Dirichlet form coincides with the law of the solution of the system of equations (without the smooth interaction term $\nabla\Phi_s$). We conjecture that this remains true with the model presented here, but we are mainly interested to use a pathwise approach, more explicit than the Dirichlet form method.

2 Skorohod type equation for a domain in $\mathbb{R}^{nd}$

Let us first do some general considerations on reflecting boundaries and associated Skorohod's problems.

2.1 Geometrical estimates on the reflecting boundary

In this subsection and in the next one, the dimension of the Euclidean space is fixed equal to an integer m.

For a domain $D \subset \mathbb{R}^m$ we define the set $\mathcal{N}_x = \mathcal{N}_x(D)$ of inward normal unit vectors at $x \in \partial D$ by

$$\mathcal{N}_x = \bigcup_{\ell>0} \mathcal{N}_{x,\ell}, \quad \mathcal{N}_{x,\ell} = \{\mathbf{n} \in \mathbb{R}^m : |\mathbf{n}| = 1, U_\ell(x-\ell\mathbf{n}) \cap D = \emptyset\}.$$

Let us recall some usual regularity conditions one can suppose on the boundary ∂D of the domain D.

Condition (A) (uniform exterior sphere condition):
There exists a constant $\alpha_0 > 0$ such that

$$\forall x \in \partial D \qquad \mathcal{N}_x = \mathcal{N}_{x,\alpha_0} \neq \emptyset.$$

This means that a small enough sphere rolling along the boundary of D reaches each point of this boundary.

Condition (B):
There exists constants $\delta_0 > 0$ and $\beta_0 \in [1,\infty)$ such that :
for any $x \in \partial D$ there exists a unit vector $\mathbf{l}_x$ verifying

$$\forall \mathbf{n} \in \bigcup_{y \in U_{\delta_0}(x)\cap\partial D} \mathcal{N}_y, \qquad \langle \mathbf{l}_x, \mathbf{n}\rangle \geqslant \frac{1}{\beta_0}.$$

For example, Condition (B) is satisfied when the domain verifies the uniform interior cone condition (see [Sai87]).

Under Condition (A), each $x \notin D$ such that $d(x,D) < \alpha_0$ has a unique projection $\overline{x}$ on ∂D, satisfying $d(x,D) = |x - \overline{x}|$ and $\frac{\overline{x}-x}{|\overline{x}-x|} \in \mathcal{N}_{\overline{x}}$. We extend this projection operator to $x \in D$ by $\overline{x} = x$.

If, for each x and y in the neighbourhood of D, the distance $|\overline{x} - \overline{y}|$ between there projections is controlled by the distance $|x-y|$ between the points, then the boundary of D is smooth in a certain sense. Saisho established some useful regularity estimates on the solutions of Skorohod equations in a domain satisfying such a smoothness condition. We will use these estimates for a domain satisfying a priori only Condition (A). So we first have to prove that :

Lemma 2.1 *If a domain D satisfies Condition* (A)*, then the projection operator on D satisfies the following continuity property :*

$$\text{(2.1)}\qquad \begin{array}{l} \textit{For all } x, y \in \mathbb{R}^m \textit{ such that } d(x,D) < \alpha_0 \textit{ and } d(y,D) < \alpha_0 \\ |\overline{x} - \overline{y}| \leqslant \left(1 + \frac{|x-\overline{x}|+|y-\overline{y}|}{2\alpha_0 - |x-\overline{x}| - |y-\overline{y}|}\right)|x-y| \end{array}$$

Proof.

Let us take $x, y \in \mathbb{R}^m$ such that $d(x,D) < \alpha_0$ and $d(y,D) < \alpha_0$.
If $x = \overline{x}$ and $y = \overline{y}$, inequality (2.1) is clearly satisfied. So, in the sequel, we assume that $(|x-\overline{x}|, |y-\overline{y}|) \neq (0,0)$. We now define c_x on the line $(x,\overline{x})$ and c_y on the line $(y,\overline{y})$ in order to have:

$$|\overline{x} - c_x| = |\overline{y} - c_y| = \alpha_0,\ x \in [c_x, \overline{x}],\ y \in [c_y, \overline{y}] \text{ and } |\overline{x} - c_y| \geqslant \alpha_0,\ |\overline{y} - c_x| \geqslant \alpha_0.$$

Such c_x and c_y always exist, just choose them as follows:

- if $x \neq \overline{x}$ let $\overline{x} - c_x = \alpha_0 \frac{\overline{x}-x}{|\overline{x}-x|}$. Since $\frac{\overline{x}-x}{|\overline{x}-x|} \in \mathcal{N}_x$, this choice implies that $U_{\alpha_0}(c_x) \cap D = \emptyset$, thus $|\overline{y} - c_x| \geqslant \alpha_0$.
- if $y \neq \overline{y}$ let $\overline{y} - c_y = \alpha_0 \frac{\overline{y}-y}{|\overline{y}-y|}$. This implies that $|\overline{x} - c_y| \geqslant \alpha_0$.
- if $x = \overline{x}$, and thus $y \neq \overline{y}$, let $c_x - \overline{x} = \alpha_0 \frac{\overline{x}-c_y}{|\overline{x}-c_y|}$.
  With this choice, $|c_x - c_y| \geqslant \alpha_0 + |\overline{x} - c_y| \geqslant 2\alpha_0$ thus $|\overline{y} - c_x| \geqslant \alpha_0$.
- if $y = \overline{y}$, and thus $x \neq \overline{x}$, let $c_y - \overline{y} = \alpha_0 \frac{\overline{y}-c_x}{|\overline{y}-c_x|}$. This choice again implies $|\overline{x} - c_y| \geqslant \alpha_0$.

Let us introduce the notations:

$$\gamma_x = \frac{|x-c_x|}{\alpha_0}, \gamma_y = \frac{|y-c_y|}{\alpha_0}, e_x = \frac{\overline{x}-c_x}{\alpha_0}, e_y = \frac{\overline{y}-c_y}{\alpha_0} \text{ and } h = \frac{c_y - c_x}{\alpha_0}$$

We have:

$$1 \geqslant \gamma_x > 0,\ 1 \geqslant \gamma_y > 0,\ |e_x| = 1,\ |e_y| = 1,\ |e_x - h| \geqslant 1 \text{ and } |e_y + h| \geqslant 1$$

and inequality (2.1) becomes:

$$|e_x - h - e_y| \leqslant \frac{2}{\gamma_x + \gamma_y} |\gamma_x e_x - h - \gamma_y e_y|.$$

It is sufficient to prove this inequality for $\gamma_x \neq \gamma_y$: the continuity of the right hand side when γ_y tends to γ_x will then prove that it holds for any (γ_x, γ_y) in $]0,1]^2$.

From now on, the parameters $\gamma_x, \gamma_y \in]0,1]$, $\gamma_x \neq \gamma_y$, are fixed. We only have to prove that the $\mathcal{C}^1$-function

$$F(e_x, e_y, h) = \frac{|e_x - h - e_y|^2}{|\gamma_x e_x - h - \gamma_y e_y|^2}$$

defined on the $\mathcal{C}^1$-manifold

$$\mathcal{V} = \{(e_x, e_y, h) \in (\mathbb{R}^m)^3 \ , \ |e_x| = 1, |e_y| = 1, |e_x - h| \geqslant 1, |e_y + h| \geqslant 1\}$$

admits $(\dfrac{2}{\gamma_x + \gamma_y})^2$ as an upper bound.

First remark that since $|e_x| = 1$

$$|e_x - h| \geqslant 1 \quad \Leftrightarrow \quad \langle e_x, h\rangle \leqslant \frac{|h|^2}{2}$$

Remark also that F is well-defined on $\mathcal{V}$, since, if $\langle e_x, h\rangle \leqslant \frac{|h|^2}{2}$,

$$\begin{aligned}
&\gamma_x e_x - h - \gamma_y e_y = 0 \\
&\quad \Rightarrow e_y = \tfrac{\gamma_x}{\gamma_y} e_x - \tfrac{1}{\gamma_y} h \text{ and } h \neq 0 \quad \text{(because } |e_x| = |e_y| \text{ and } \gamma_x \neq \gamma_y\text{)} \\
&\quad \Rightarrow |e_y + h|^2 = 1 + 2\tfrac{\gamma_x}{\gamma_y}\langle e_x, h\rangle - 2\tfrac{1}{\gamma_y}\langle h, h\rangle + |h|^2 \\
&\qquad\qquad\qquad\quad \leqslant 1 + \tfrac{\gamma_x}{\gamma_y}|h|^2 - \tfrac{2}{\gamma_y}|h|^2 + |h|^2 = 1 - \tfrac{2-\gamma_x-\gamma_y}{\gamma_y}|h|^2 < 1
\end{aligned}$$

Using the famous theorem about differentiable functions on manifolds, we obtain

$$\sup_{\mathcal{V}} F = \max(\sup_{\mathcal{V}_\infty} F, \sup_{\mathcal{V}_\nabla} F, \sup_{\mathcal{V}_\partial} F)$$

where

$$\begin{aligned}
\mathcal{V}_\infty &= \{(e_x, e_y, h) \in \mathcal{V}, \ |h| \geqslant 4\} \\
\mathcal{V}_\nabla &= \{(e_x, e_y, h) \in \mathcal{V}, \ |e_x - h| > 1, |e_y + h| > 1, \nabla F \in \mathrm{Span}(\nabla(|e_x|^2 - 1), \nabla(|e_y|^2 - 1))\} \\
\mathcal{V}_\partial &= \{(e_x, e_y, h) \in \mathcal{V}, \ |e_x - h| = 1 \text{ or } |e_y + h| = 1\}.
\end{aligned}$$

The computation of an upper bound for F on $\mathcal{V}_\infty$ is very easy. Just use the triangular inequality twice:

$$\begin{aligned}
&|e_x - h - e_y| \leqslant (1 - \gamma_x)|e_x| + |\gamma_x e_x - h - \gamma_y e_y| + (1 - \gamma_y)|e_y| \\
&|\gamma_x e_x - h - \gamma_y e_y| \geqslant |h| - \gamma_x - \gamma_y \geqslant 2 \text{ if } |h| \geqslant 4
\end{aligned}$$

thus

$$\sup_{\mathcal{V}_\infty} F \leqslant \left(1 + \frac{2 - \gamma_x - \gamma_y}{2}\right)^2 \leqslant \left(\frac{\gamma_x + \gamma_y}{\gamma_x + \gamma_y} + \frac{2 - \gamma_x - \gamma_y}{\gamma_x + \gamma_y}\right)^2 \ = \ \left(\frac{2}{\gamma_x + \gamma_y}\right)^2.$$

To compute an upper bound for F on $\mathcal{V}_\nabla$, we remark that $\nabla F = (\nabla_{e_x} F, \nabla_{e_y} F, \nabla_h F)$ and $\nabla(|e_x|^2 - 1) = (2e_x, 0, 0)$, $\nabla(|e_y|^2 - 1) = (0, 2e_y, 0)$.

If $\nabla F \in \mathrm{Span}(\nabla(|e_x|^2-1), \nabla(|e_y|^2-1))\}$ then $\nabla_h F = 0$, that is:

$$\nabla_h F = \frac{-2(e_x - h - e_y)|\gamma_x e_x - h - \gamma_y e_y|^2 + 2(\gamma_x e_x - h - \gamma_y e_y)|e_x - h - e_y|^2}{|\gamma_x e_x - h - \gamma_y e_y|^4} = 0.$$

This implies that

$$|e_x - h - e_y| \ |\gamma_x e_x - h - \gamma_y e_y|^2 = |\gamma_x e_x - h - \gamma_y e_y| \ |e_x - h - e_y|^2$$

which exactly means that $F(e_x, e_y, h) = \sqrt{F(e_x, e_y, h)}$, i.e. $F(e_x, e_y, h)$ equals 0 or 1. Thus $\sup\limits_{\mathcal{V}_\nabla} F \leqslant 1$.

Finally, we compute a bound for F on $\mathcal{V}_\partial$. We will compute an upper bound for $F(e_x, e_y, h)$ when $|e_x| = |e_y| = 1$, $|e_x - h| \geqslant 1$ and $|e_y + h| = 1$. The computation for $|e_x - h| = 1$ and $|e_y + h| \geqslant 1$ is exactly the same (just exchange e_x and e_y, γ_x and γ_y and replace h by $-h$).

If $|e_x| = |e_y| = |e_y + h| = 1$ and $|e_x - h| \leqslant 1$:

$$\begin{array}{lll} |e_y + h| = 1 & \Leftrightarrow \quad 2\langle e_y, h\rangle = -|h|^2 & \Leftrightarrow \quad 2\langle e_y + h, h\rangle = |h|^2 \\ |e_x - h| \geqslant 1 & \Leftrightarrow \quad -2\langle e_x, h\rangle \geqslant -|h|^2 & \end{array}$$

thus

$$\begin{aligned} &|\gamma_x e_x - h - \gamma_y e_y|^2 \\ &\quad = \ |\gamma_x e_x - \gamma_y(e_y + h) - (1-\gamma_y)h|^2 \\ &\quad = \ \gamma_x^2 + \gamma_y^2 - 2\gamma_x\gamma_y\langle e_x, e_y + h\rangle + (1-\gamma_y)|h|^2 - 2\gamma_x(1-\gamma_y)\langle e_x, h\rangle \\ &\quad \geqslant \gamma_x^2 + \gamma_y^2 - 2\gamma_x\gamma_y\langle e_x, e_y + h\rangle \end{aligned}$$

and since $|e_x - h - e_y|^2 = 2 - 2\langle e_x, e_y + h\rangle$, we obtain :

$$F(e_x, e_y, h) \leqslant \frac{2 - 2\langle e_x, e_y + h\rangle}{\gamma_x^2 + \gamma_y^2 - 2\gamma_x\gamma_y\langle e_x, e_y + h\rangle}.$$

An elementary derivative computation prove that, when $A \geqslant B$, the function $u \to \frac{2-2u}{A-Bu}$ decreases on $[-1;1]$, thus $\sup_{[-1;1]} \frac{2-2u}{A-Bu} = \frac{4}{A+B}$ and

$$\sup_{\mathcal{V}_\partial} F \leqslant \frac{4}{\gamma_x^2 + \gamma_y^2 + 2\gamma_x\gamma_y} = (\frac{2}{\gamma_x + \gamma_y})^2.$$

The proof is complete. ■

2.2 Regularity estimates for the solution of Skorohod's problem

Let D be a domain of $\mathbb{R}^m$. For a given $w \in W_0(\mathbb{R}^m) = \{w \in W(\mathbb{R}^m) : w(0) = 0\}$ and $x \in \overline{D}$, we consider the following Skorohod equation with reflecting boundary ∂D :

$$\zeta(t) = x + w(t) + \varphi(t), \quad t \geqslant 0. \tag{2.2}$$

A solution is a pair (ζ, φ) satisfying (2.2) and the following two conditions (2.3) and (2.4) (we also call ζ a solution of (2.2)) :

(2.3) $\quad \zeta \in W(\overline{D})$.

(2.4) $\quad \varphi$ is an $\mathbb{R}^m$-valued continuous function with bounded variation on each finite time interval satisfying $\varphi(0) = 0$ and

$$\varphi(t) = \int_0^t \mathbf{n}(s) d\|\varphi\|_s, \quad \|\varphi\|_t = \int_0^t \mathbb{I}_{\partial D}(\zeta(s)) d\|\varphi\|_s,$$

where $\mathbf{n}(s) \in \mathcal{N}_{\zeta(s)}$ if $\zeta(s) \in \partial D$, and $\|\varphi\|_t$ denotes the total variation of φ on $[0, t]$.

The existence and uniqueness of solutions of Skorohod type equations were studied by many authors (Tanaka [Tan79], Lions and Sznitman [LS84], Saisho [Sai87]). Saisho (see [Sai87] theorem 4.1) proved that, under Conditions (A) and (B), Skorohod equation (2.2) admits a unique solution. Furthermore, this solution satisfies the following Lipschitz continuity property as a function of $w(\cdot)$ and x :

Lemma 2.2 *Suppose that the domain D satisfies Conditions* (A) *and* (B) *and let $\zeta(.)$ (respectively $\zeta'(.)$) be the unique solution of Skorohod equation (2.2) (resp. for $w' \in W_0(\mathbb{R}^m), x' \in \overline{D}, \quad \zeta'(t) = x' + w'(t) + \varphi'(t), \quad t \geqslant 0$).*

Then there exists a constant C_1, depending only on D, such that for each $t \geqslant 0$,

$$|\zeta(t) - \zeta'(t)| \leqslant \Big(\|w - w'\|_t + |x - x'| \Big) \exp \Big(C_1 (\|\varphi\|_t + \|\varphi'\|_t) \Big). \tag{2.5}$$

Proof.

In Proposition 4.1 of [Sai87], Saisho proved this Lipschitz continuity property under Condition (A), Condition (B), and the following additional condition on the projection operator $x \longrightarrow \overline{x}$ (called Condition (D) in [Sai87]) :

there exists $C_1 \geqslant 0$ and $C_2 \in]0, \alpha_0[$ such that for all $x, y \in \mathbb{R}^m$:

$$\max(|x - \overline{x}|, |y - \overline{y}|) \leqslant C_2 \Longrightarrow |\overline{x} - \overline{y}| \leqslant \Big(1 + C_1 \max(|x - \overline{x}|, |y - \overline{y}|) \Big) |x - y|.$$

Thanks to Lemma 2.1, the projection always have this property (for any $C_2 \in]0, \alpha_0[$ and $C_1 = 1/(\alpha_0 - C_2)$) when D satisfies Condition (A). ∎

Remark that the constant C_1 in the above Lemma a priori depends on the space dimension m.

The following lemma gives an estimate of the total variation $\|\varphi\|_t$ of the process $\varphi(t)$ (See Theorem 4.2 in [Sai87]).

Lemma 2.3 *Suppose that the domain D satisfies Conditions* (A) *and* (B)*. Then, for any finite $T > 0$, we have*

$$\|\varphi\|_t \leqslant f(\Delta_{0,T,\cdot}(w), \sup_{s \leqslant t} |w(s)|) \quad \textit{for all } 0 \leqslant t \leqslant T,$$

where f is a function defined on $W_0(\mathbb{R}^+)\times\mathbb{R}^+$ depending only on the constants $\alpha_0, \beta_0, \delta_0$ in Conditions (A) *and* (B)*, and $\Delta_{0,T,.}(w)$ denotes the modulus of continuity of w in $[0,T]$ defined as usually by*

$$\Delta_{0,T,\delta}(w) = \sup_{\substack{0<s<t<T \\ |t-s|\leqslant\delta}} |w(t) - w(s)|. \tag{2.6}$$

Moreover, the functional $w \longrightarrow f(\Delta_{0,T,.}(w), \sup_{s\leqslant t}|w(s)|)$ is bounded on each set of paths $\mathcal{W}$ satisfying $\lim_{\delta\to 0}\sup_{w\in\mathcal{W}}\Delta_{0,T,\delta}(w) = 0$.

2.3 Application to a system of finitely many hard balls

Now the dimension of the state space is $m = nd$. Let us define a system of n hard balls moving in $\mathbb{R}^d$ and reflected on the boundary of a domain $D_n \subset \mathbb{R}^{nd}$:

$$D_n = \{\mathbf{x}_n = (x_1, \dots, x_n) \in \mathbb{R}^{nd} : |x_i - x_j| > r, i \neq j\}.$$

Saisho and Tanaka [ST86] checked that for each $n \in \mathbb{N}$, the domain D_n satisfies Conditions (A) and (B).

Let $\mathbf{b} = (b_1, \dots, b_n)$ be a Lipschitz continuous $\mathbb{R}^{nd}$-valued function defined on $\mathbb{R}^{nd}$. Let also take $\mathbf{w} = (w_1, w_2, \dots) \in W_0(\mathbb{R}^d)^n$.

Saisho and Tanaka [ST86] proved that the following system of n equations in $\mathbb{R}^d$ (2.7),(2.8),(2.9) has a unique solution $\zeta = (\zeta_i)_{i=1,2,\dots,n}$, since it can be considered as a Skorohod equation in $\mathbb{R}^{nd}$ with ∂D_n as reflecting boundary :

$$\begin{aligned} &\forall i \in \{1, 2, \dots, n\} \\ &\zeta_i(t) = x_i + w_i(t) + \int_0^t b_i(\zeta(s))ds + \sum_{j=1}^n \int_0^t (\zeta_i(s) - \zeta_j(s))d\rho_{ij}(s). \end{aligned} \tag{2.7}$$

(2.8) $\quad(\zeta_i)_{i=1,2,\dots,n}$ are continuous functions with $|\zeta_i(t) - \zeta_j(t)|\geqslant r$, $t \in [0,\infty)$, $i \neq j$.
(2.9) $\quad(\rho_{ij})_{i,j=1,2,\dots,n}$ are continuous nondecreasing functions with $\rho_{ij}(0) = 0$, $\rho_{ij} \equiv \rho_{ji}$ and

$$\rho_{ij}(t) = \int_0^t \mathbb{1}_{\{r\}}(|\zeta_i(s) - \zeta_j(s)|)d\rho_{ij}(s).$$

From now on, the number n of interacting hard balls we study becomes random but remains a.s. finite. To study such systems, we introduce the new configuration space $\mathcal{D}$:

$$\mathcal{D} = \bigcup_{n=0}^{\infty} \overline{D_n}$$

where $\overline{D_0} = \{\emptyset\}$, $\overline{D_1} = \mathbb{R}^d$, and for $n\geqslant 2$,

$$\overline{D_n} = \{\mathbf{x}_n = (x_1, x_2, \dots, x_n) \in \mathbb{R}^{nd} : |x_i - x_j|\geqslant r, 1\leqslant i < j\leqslant n\}.$$

Let Ψ be a function on $\{\emptyset\} \cup (\bigcup_{n=1}^{\infty} \mathbb{R}^{nd})$ satisfying $\Psi(\emptyset) = 0$ and the following conditions:

$(\Psi.1)$ Ψ is a $\mathcal{C}^1$ – function, invariant by permutation on $(\mathbb{R}^d)^n$ for each $n \geqslant 1$,

with $\nabla \Psi$ Lipschitz continuous.

$(\Psi.2)$ $\exists K_\Psi \in \mathbb{R},\ \forall n \in \mathbb{N},\ \inf_{(x_1,\cdots,x_n,y)\in \overline{D_{n+1}}} \big(\Psi(x_1,\cdots,x_n,y) - \Psi(x_1,\cdots,x_n)\big) \geqslant K_\Psi$

$(\Psi.3)$ $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\mathbb{R}^{nd}} \chi_h(\mathbf{x}_n) \exp(-\Psi(\mathbf{x}_n)) d\mathbf{x}_n < +\infty,$

where $\chi_h(\mathbf{x}_n) = \exp(-\sum_{1\leqslant i<j\leqslant n} \Phi_h(x_i, x_j)) = \mathbb{1}_{\{x_1,x_2,\cdots,x_n\}\in\mathfrak{X}}$.

We define a probability measure μ_z^Ψ on $\mathcal{D}$ by $\mu_z^\Psi(\{\emptyset\}) = \frac{1}{Z_z^\Psi}$ and

(2.10) $$\mu_z^\Psi(A) = \frac{1}{Z_z^\Psi} \frac{z^n}{n!} \int_A \chi_h(\mathbf{x}_n) \exp(-\Psi(\mathbf{x}_n)) d\mathbf{x}_n, \text{ for any Borel set } A \subset \overline{D_n}$$

where $d\mathbf{x}_n = dx_1 dx_2 \dots dx_n$ and $Z_z^\Psi = 1 + \sum_{n=1}^{\infty} \frac{z^n}{n!} \int_{\mathbb{R}^{nd}} \chi_h(\mathbf{x}_n) \exp(-\Psi(\mathbf{x}_n)) d\mathbf{x}_n$.

By the symmetry property of Ψ, it is clear that μ_z^Ψ can be considered as a Probability measure on $\mathfrak{X}$.

For $\mathbf{x} \in \mathcal{D}$ and $\mathbf{w} = (w_1, w_2, \dots) \in W_0(\mathbb{R}^d)^{\mathbb{N}}$, we put

(2.11) $$\zeta^\Psi(t, \mathbf{x}, \mathbf{w}) = \begin{cases} \zeta^\Psi(t, \mathbf{x}_n, \mathbf{w}_n), & \text{if } \mathbf{x} = \mathbf{x}_n, n \in \mathbb{N}, \\ 0, & \text{if } \mathbf{x} = \emptyset, \end{cases}$$

where $\mathbf{w}_n = (w_1, w_2, \dots, w_n)$, and $\zeta^\Psi(t, \mathbf{x}_n, \mathbf{w}_n)$ is the unique solution of the equation (2.7) where the drift $\mathbf{b}$ is given by

(2.12) $$\mathbf{b}(\mathbf{x}_n) = -\nabla \Psi(\mathbf{x}_n), n \in \mathbb{N}.$$

We denote by P_W the Wiener measure on $W_0(\mathbb{R}^d)$. As in [Tan96] Lemma 2.4, we compute a bound for the probability that ζ^Ψ oscillate too much when $\mathbf{w}$ is a Brownian motion and the initial law is Gibbsian :

Lemma 2.4 *The process $\zeta^\Psi(t, ., .)$ is a reversible diffusion process under the probability $\mu_z^\Psi \otimes P_W^{\otimes \mathbb{N}}$. Moreover, for any finite time $T > 0$, there exists positive constants C_3 and C_4 depending only on T and Ψ such that*

$$\forall \ell \in \mathbb{N}, \forall \varepsilon, \delta > 0$$
$$\mu_z^\Psi \otimes P_W^{\otimes \mathbb{N}}\big(\exists i \in \mathbb{N} \text{ s.t. } \Delta_{0,T,\delta}(\zeta_i^\Psi) \geqslant \varepsilon \text{ and } \zeta_i^\Psi(0) \in U_\ell\big) \leqslant C_3 z |U_\ell| \exp(-C_4 \tfrac{\varepsilon^2}{\delta})$$

We now need to control the geometrical repartition of the particles in $\mathbb{R}^d$. To this aim, we introduce the concept of cluster.

For $\eta \in \mathfrak{X}, r' > r$ and two points x, y in $\mathbb{R}^d$, we say that a continuous curve γ is a r'-connection between x and y with respect to (η, r') if $x, y \in \gamma$ and $\gamma \subset U_{\frac{r'}{2}}(\eta)$. Then the occupied cluster $\mathbf{C}(r', x, \eta)$ of x is defined by

$$\mathbf{C}(r', x, \eta) = \{y \in \eta : \exists \text{ an occupied connection between } x \text{ and } y\ \}.$$

The set $U_{\frac{r'}{2}}(\mathbf{C}(r', x, \eta))$ is the connected component of $U_{\frac{r'}{2}}(\eta)$ containing x.

First we show the following estimate on the cardinal of the set $\mathbf{C}(r', x, \eta)$.

Lemma 2.5 *Let μ_z^Ψ be the probability measure on $\mathcal{D}$ introduced in (2.10). Then, for any $M \in \mathbb{R}^+$, there exists a constant $C_5 = C_5(r, d, z)$ such that, for any $\ell \in \mathbb{N}^*$ and $0 < \varepsilon < 1$,*

$$\mu_z^\Psi\Big(\exists x \in U_\ell, \sharp\mathbf{C}(r+\varepsilon, x, .) > M^d\Big) \leqslant C_5 \ell^d \varepsilon^{[\frac{rM}{2r+2}]} \exp(-[\frac{rM}{2r+2} + 1] K_\Psi).$$

Proof. A set of diameter ℓ cannot contain more than $(\ell/r)^d$ hard balls of diameter r. Therefore if $\sharp\mathbf{C}(r+\varepsilon, x, \eta) > M^d$ then the diameter of $U_{\frac{r+\varepsilon}{2}}(\mathbf{C}(r+\varepsilon, x, \eta))$ is larger than rM, and this in turn implies the existence of $\{y_1, \cdots, y_{M'}\} \subset \eta$ such that $|y_1| \leqslant \ell + \frac{r+\varepsilon}{2}$, $|y_1 - y_2| \leqslant r + \varepsilon$, $\dots$ $|y_{M'-1} - y_{M'}| \leqslant r + \varepsilon$ for some $M' = [\frac{rM}{2(r+\varepsilon)}] + 1$.

$$\begin{aligned}
&\mu_z^\Psi\Big(\{\eta \text{ such that } \exists x \in U_\ell, \sharp\mathbf{C}(r+\varepsilon, x, \eta) > M^d\}\Big) \\
&\leqslant \mu_z^\Psi\Big(\{\eta \text{ such that } \exists \{y_1, \cdots, y_{M'}\} \subset \eta, \\
&\qquad\qquad |y_1| \leqslant \ell + \frac{r+\varepsilon}{2},\ |y_1 - y_2| \leqslant r+\varepsilon,\ \dots,\ |y_{M'-1} - y_{M'}| \leqslant r+\varepsilon\}\Big) \\
&\leqslant \exp(-M' K_\Psi) \frac{1}{Z_z^\Psi} \sum_{n=M'}^{\infty} \frac{z^n}{n!} \binom{n}{M'} (M')! \\
&\qquad\qquad \times \int_{\mathbb{R}^{(n-M')d}} \chi_h(\mathbf{x}_{n-M'}) \exp(-\Psi(\mathbf{x}_{n-M'})) d\mathbf{x}_{n-M'} \\
&\qquad\qquad \times \int_{U_{\ell + \frac{r+\varepsilon}{2}}} dy_1 \int_{U_{r+\varepsilon}(y_1)} dy_2 \cdots \int_{U_{r+\varepsilon}(y_{M'-1})} \chi_h(\mathbf{y}_{M'}) dy_{M'} \\
&\leqslant \exp(-M' K_\Psi) |U_{\ell + \frac{r+\varepsilon}{2}}| |U_{r+\varepsilon} \setminus U_r|^{M'-1} z^{M'} \\
&\leqslant C_5 \ell^d \varepsilon^{M'-1} z^{M'} \exp(-M' K_\Psi)
\end{aligned}$$

for some integer $M' \geqslant \frac{rM}{2r+2}$ and with C_5 a constant depending only on r, d and z. This completes the proof. ■

We now define a set of regular paths, in the sense that their modulus of continuity is small enough and, at each step of a time partition, the size of the clusters is bounded. Let $\varepsilon_2 > \varepsilon_1 > 0$, $\delta > 0$ and $T, \ell \in \mathbb{N}$. We denote by $\Lambda(\varepsilon_1, \varepsilon_2, \delta, T, M, \ell)$ the set of all elements $\xi = \{\xi_i(\cdot)\}_i \in W(\mathfrak{X})$ satisfying

(2.13)	$\forall i \in \bigcup_{t \in [0,T]} \mathbf{J}(\xi(t), U_\ell)$,	$\Delta_{0,T,\delta}(\xi_i(\cdot)) \leqslant \varepsilon_1$,	
(2.14)	$\forall x \in U_\ell$,	$\forall k = 0, 1, \dots, [T/\delta]$,	$\sharp\mathbf{C}(r+\varepsilon_2, x, \xi(k\delta)) \leqslant M$.

Remark that if $\xi(\cdot) \in \Lambda(\varepsilon_1, \varepsilon_2, \delta, T, M, \ell)$ and $\varepsilon_2 > 2\varepsilon_1$, then

$$\forall x \in U_\ell, \quad \forall t \in [0,T], \qquad \sharp \mathbf{C}(r + \varepsilon_2 - 2\varepsilon_1, x, \xi(t)) \leqslant M.$$

We then obtain the following lemma :

Lemma 2.6 *Let $0 < \kappa_2 < \kappa_1 < \frac{1}{2}$, $z > 0$ and $T, p \in \mathbb{N}$. Then for any $\beta > 0$ we can choose $M = M(\kappa_1, \kappa_2, T, p) \in \mathbb{N}$ and $C_6 = C_6(\kappa_1, \kappa_2, z, T, p) > 0$ such that*

$$\forall m \in \mathbb{N}, \quad \mu_z^\Psi \otimes P_W^{\otimes \mathbb{N}}(\zeta^\Psi \in \Lambda(m^{-\kappa_1}, m^{-\kappa_2}, \frac{1}{m}, T, M, m^p)^c) \leqslant C_6 m^{-\beta}.$$

Proof. It is a consequence of Lemmas 2.4 and 2.5. ■

3 Approximation of the solution and convergence

Let $\mathbf{J}$ be any nonempty finite subset of $\mathbb{N}$. We now consider an infinite system of particles in which only a finite number (those indexed by $\mathbf{J}$) move following the dynamics defined in (2.7). Let $\mathbf{b} = (b_i)_{i \in \mathbf{J}}$ be an $(\mathbb{R}^d)^{\mathbf{J}}$-valued Lipschitz continuous function defined on $(\mathbb{R}^d)^{\mathbf{J}}$, $\mathbf{x} = (x_1, x_2, \dots)$ such that $\{x_1, x_2, \dots\} \in \mathfrak{X}$ and $\mathbf{w} = (w_1, w_2, \dots) \in W_0(\mathbb{R}^d)^{\mathbb{N}}$. We then obtain the following system of equations (3.1) under the conditions (3.2) and (3.3):

$$(3.1) \qquad \xi_i(t) = \begin{cases} x_i + w_i(t) + \displaystyle\int_0^t b_i(\xi(s))ds + \sum_{j \in \mathbf{J}} \int_0^t (\xi_i(s) - \xi_j(s)) d\rho_{ij}(s) & \text{if } i \in \mathbf{J}, \\ x_i & \text{if } i \notin \mathbf{J}. \end{cases}$$

(3.2) $(\xi_i)_{i \in \mathbf{J}}$ are continuous functions with $|\xi_i(t) - \xi_j(t)| \geqslant r$, $t \in [0, \infty)$, $i \neq j$.
(3.3) $(\rho_{ij}, i, j \in \mathbf{J})$ are continuous nondecreasing functions with $\rho_{ij}(0) = 0$, $\rho_{ij} \equiv \rho_{ji}$ and

$$\rho_{ij}(t) = \int_0^t \mathbb{1}_{\{r\}}(|\xi_i(s) - \xi_j(s)|) d\rho_{ij}(s).$$

For $i \notin \mathbf{J}$ or $j \notin \mathbf{J}$, $\rho_{ij} \equiv 0$.

Existence and uniqueness of the solution of (3.1) were discussed in the previous section.

Let Φ_s be the smooth pair potential with infinite range defined in section 1.2, and let $\psi^{\ell,\eta}, \ell \in \mathbb{N}, \eta \in \mathfrak{X}$ be nonnegative smooth functions defined on $\mathbb{R}^d$ with the following properties :

(3.4) $\nabla \psi^{\ell,\eta}$ is bounded Lipschitz continuous

(3.5) $\psi^{\ell,\eta} = 0$ on $U_\ell^\eta = U_\ell \setminus U_r(\eta \cap U_\ell^c)$

$$(3.6) \qquad \sum_{\ell \in \mathbb{N}} \sup_{\eta \in \mathfrak{X}} \int_{\mathbb{R}^d \setminus U_\ell^\eta} \exp\big(-\psi^{\ell,\eta}(x)\big) dx \quad < \quad +\infty.$$

Such functions obviously exist: take for example $\psi^{\ell,\eta}(x) = l^{d+1}\delta_\eta(x)$ with δ_η a $\mathcal{C}^2$-function with bounded derivatives which is equivalent to $d(.,U_\ell^\eta)$ on $\mathbb{R}^d$ (see [Ste70] p.171).

We can now define on $\bigcup_{n=1}^\infty \mathbb{R}^{nd}$ the following potential, as perturbation by the self potential $\psi^{\ell,\eta}$ of the smooth pair potential Φ_s, with η as fixed external configuration : for any $\mathbf{J}$ finite subset of $\mathbb{N}$,

$$\Psi^{\ell,\eta}(\mathbf{x_J}) = \sum_{i\in\mathbf{J}} \psi^{\ell,\eta}(x_i) + \sum_{\substack{i,j\in\mathbf{J}\\ i<j}} \Phi_s(x_i - x_j) + \sum_{\substack{i\in\mathbf{J}\\ j,|\eta_j|\geqslant\ell}} \Phi_s(x_i-\eta_j). \tag{3.7}$$

Note that $\Psi^{\ell,\eta}$ satisfies the assumptions made on the function Ψ in the section 2.3 : $(\Psi.1)$ is obvious, $(\Psi.2)$ is true with $K_\Psi = -2\overline{\Phi_s}$ and $(\Psi.3)$ comes from (3.6).

From now on, and for the rest of this section, let $\mathbf{X}_0 = \{X_1, X_2, \dots\}$ be a fixed $\mathfrak{X}$-valued random variable with Gibbsian law $\mu \in \mathcal{G}(z,\Phi)$. Let $(B_i(t), i \in \mathbb{N})$ be a family of independent $\mathbb{R}^d$-valued Brownian motions.
We now consider, for each $\ell \in \mathbb{N}$, a particular case of equation (3.1) with $x_i = X_i, w_i = B_i(.), b_i = -\frac{1}{2}\nabla_i\Psi^{\ell,\mathbf{X}_0}$ and $\mathbf{J} = \mathbf{J}(\mathbf{X}_0, U_\ell) = \{i \in \mathbb{N} : |X_i| < \ell\}$ random. The unique solution of this equation is denoted by $(\mathbf{X}^\ell(t), \mathbf{L}^\ell(t)) = (X_i^\ell(t), L_{ij}^\ell(t), i,j \in \mathbb{N})$ and satisfies :

$$X_i^\ell(t) = \begin{cases} X_i + B_i(t) - \frac{1}{2}\int_0^t \nabla_i \Psi^{\ell,\mathbf{X}_0}(\mathbf{X}_\mathbf{J}^\ell(s))ds & \\ \qquad\qquad +\sum_{j\in\mathbf{J}(\mathbf{X}_0,U_\ell)} \int_0^t (X_i^\ell(s) - X_j^\ell(s))dL_{ij}^\ell(s) & \text{if } i \in \mathbf{J}(\mathbf{X}_0,U_\ell) \\ X_i & \text{if } i \notin \mathbf{J}(\mathbf{X}_0,U_\ell). \end{cases} \tag{3.8}$$

(3.9) $(X_i^\ell(.))_{i\in\mathbf{J}(\mathbf{X}_0,U_\ell)}$ are continuous processes with $|X_i^\ell(t) - X_j^\ell(t)| \geqslant r$, $\forall t \geqslant 0$, $i \neq j$.
(3.10) $(L_{ij}^\ell, i,j \in \mathbb{N})$ are continuous nondecreasing processes with $L_{ij}^\ell(0) = 0$, $L_{ij}^\ell \equiv L_{ji}^\ell$ and

$$L_{ij}^\ell(t) = \begin{cases} \int_0^t \mathbb{I}_{\{r\}}(|X_i^\ell(s) - X_j^\ell(s)|)dL_{ij}^\ell(s) & \text{if } i,j \in \mathbf{J}(\mathbf{X}_0,U_\ell) \\ 0 & \text{otherwise.} \end{cases}$$

Then we have the following convergence result :

Proposition 3.1 *The sequence of processes $(\mathbf{X}^\ell)_{\ell\in\mathbb{N}^*}$ converges a.s. in $W(\mathfrak{X})$ to a reversible process $\mathbf{X}^\infty$ with values in $\mathcal{C}\cap\Theta$, where*

$$\Theta = \bigcap_{0<\kappa<\frac{1}{2}} \bigcap_{p=1}^\infty \bigcap_{T=1}^\infty \bigcup_{m_0=1}^\infty \bigcap_{m=m_0}^\infty \Theta[m^{-\kappa}, \frac{1}{m}, T, m^p], \qquad \textit{and}$$

$$\Theta[\varepsilon,\delta,T,\ell] = \{\{\xi_i\} \in W(\mathfrak{X}) : \Delta_{0,T,\delta}(\xi_i) < \varepsilon \textit{ for any } i \textit{ with } \min_{t\in[0,T]} |\xi_i(t)| < \ell\}.$$

Moreover, the process $\mathbf{X}^\infty(.)$ is the unique solution of the infinite dimensional system of stochastic equations (1.12) *when the initial condition is equal to* $\mathbf{X}_0 = \{X_1, X_2, \dots\}$.

The rest of this section is devoted to the proof of Proposition 3.1. We first construct a set of probability one on which $(\mathbf{X}^\ell)$ is a Cauchy sequence and then we prove that the limit point is the unique solution of (1.12).

Lemma 3.2 *Let $0 < \kappa_2 < \kappa_1 < \frac{1}{2}$ and $T, p \in \mathbb{N}$. Then we can choose $M \in \mathbb{N}$ such that*

$$\sum_{m=1}^{\infty} \sum_{\ell=m^p}^{(m+1)^p} P(\mathbf{X}^\ell(\cdot) \in \Lambda(m^{-\kappa_1}, m^{-\kappa_2}, \frac{1}{m}, T, M, m^p)^c) < +\infty.$$

Proof. Let $\mu_z^{\ell,\eta}$ be the Gibbs measure defined by (2.10) where the function Ψ is taken equal to the potential $\Psi^{\ell,\eta}$ defined in (3.7). By comparing $\mu_z^{\ell,\eta}$ and the local specification $\mu_{U_\ell,\eta,z}$ (defined by (1.11)) as detailed in the Proposition 6.1 (steps 1 and 2) in [FR], we obtain

$$\mu_z^{\ell,\eta}(N_{(U_\ell^\eta)^c} \neq 0) \leqslant \int_{\mathfrak{X}} N_{(U_\ell^\eta)^c}(\xi) \mu_z^{\ell,\eta}(d\xi) \leqslant z \exp\big(-2\overline{\Phi_s}\big) \int_{\mathbb{R}^d \setminus U_\ell^\eta} \exp\big(-\psi^{\ell,\eta}(x)\big) dx;$$

the same upper-bound holds for $|\mu_z^{\ell,\eta} - \mu_{U_\ell,\eta,z}|(N_{(U_\ell^\eta)^c} = 0)$, which leads to the estimate

$$\|\mu_z^{\ell,\eta} - \mu_{U_\ell,\eta,z}\| \leqslant 2z \exp\big(-2\overline{\Phi_s}\big) \int_{\mathbb{R}^d \setminus U_\ell^\eta} \exp\big(-\psi^{\ell,\eta}(x)\big) dx$$

where $\|\nu\|$ denotes the total variation of a signed measure ν. Then,

$$\begin{aligned} & P\big(\mathbf{X}^\ell(\cdot) \in \Lambda(m^{-\kappa_1}, m^{-\kappa_2}, \tfrac{1}{m}, T, M, m^p)^c\big) \\ & \leqslant \int P\big(\mathbf{X}^\ell(\cdot) \in \Lambda(m^{-\kappa_1}, m^{-\kappa_2}, \frac{1}{m}, T, M, m^p)^c | \mathbf{X}^\ell(0) = \xi\big) d\mu_z^{\ell,\eta}(\xi) d\mu(\eta) \\ & \quad + \int \|\mu_z^{\ell,\eta} - \mu_{U_\ell,\eta,z}\| d\mu(\eta). \end{aligned}$$

By Lemma 2.6, assumption (3.6) and the above inequalities the series in Lemma 3.2 converges. ■

We fix the parameters $0 < \kappa_2 < \kappa_1 < \frac{1}{2}$, $T \in \mathbb{N}$ and $p \in \mathbb{N}$.
By the scaling property of the Brownian motion $\mathbf{B}$ and Doob's inequality, we control the modulus of continuity of $\mathbf{B}$ as follows :

$$\sum_{m=1}^{\infty} \sum_{\ell=m^p}^{(m+1)^p} P(\Delta_{0,T,1/m}(B_i) > m^{-\kappa_1} \text{ for some } i \text{ with } \min_{t \in [0,T]} |X_i^\ell(t)| < m^p) < +\infty.$$

Combining this and Lemma 3.2, by Borel Cantelli's Lemma, for almost all ω, there exists $M \in \mathbb{N}$ and $m_0 = m_0(\omega) \in \mathbb{N}$ such that

for $m \geqslant m_0$ and $m^p \leqslant \ell < (m+1)^p, \mathbf{X}^\ell(\cdot),\ \mathbf{X}^{\ell+1}(\cdot) \in \Lambda(m^{-\kappa_1}, m^{-\kappa_2}, \frac{1}{m}, T, M, m^p)$
and $\forall i \in \bigcup_{t \in [O,T]} \mathbf{J}(\mathbf{X}^\ell(t), U_{m^p})$, $\quad \forall h \in]0,T]$, $\quad \Delta_{0,T,h}(B_i) \leqslant 2h^{\kappa_1}$.

We are now looking for an upper bound for $|X_i^\ell - X_i^{\ell+1}|$ when i belongs to some subset of indices.

To this aim we need a comparison lemma formulated under the following generality : for $\alpha = 1, 2$, let $\mathbf{x}^{(\alpha)} \in \mathfrak{X}$, and $\mathbf{J}(\alpha)$ be finite subsets of $\mathbb{N}$. We also define two drift functions on $(\mathbb{R}^d)^{\mathbf{J}(\alpha)}$ by :

$$b_i^{(\alpha)}(\mathbf{x}) = -\frac{1}{2} \sum_{j \in \mathbf{J}(\alpha)} \nabla \Phi_s(x_i - x_j) + c_i^{(\alpha)}(\mathbf{x}),\ i \in \mathbf{J}(\alpha),$$

where $\mathbf{c}^{(\alpha)} = (c_i^{(\alpha)})_{i \in \mathbf{J}(\alpha)}$ is an $(\mathbb{R}^d)^{\mathbf{J}(\alpha)}$-valued Lipschitz continuous function. We denote by $(\xi^{(\alpha)}(t), \rho^{(\alpha)}(t))$ the unique solution of (3.1) with $\mathbf{J} = \mathbf{J}(\alpha)$, $\mathbf{x} = \mathbf{x}^{(\alpha)}$, $b_i = b_i^{(\alpha)}$ and $\mathbf{w} \in W_0(\mathbb{R}^d)^{\mathbb{N}}$ fixed not depending on α.

Lemma 3.3 *Suppose that there exists $R, R_0, R_1 > 0$, $M \in \mathbb{N}^*$, $\varepsilon_0 \geqslant 0$, $\varepsilon_1, \varepsilon_2 > 0$, $0 < \delta \leqslant T$ such that*

$$\begin{aligned}
&(3.11) \qquad |c_i^{(1)}(\mathbf{x})|, |c_i^{(2)}(\mathbf{x})| \leqslant g(R) \text{ if } x_i \in U_{R_1 - R} \\
&(3.12) \qquad \forall i \in \mathbf{J}(\mathbf{x}^{(1)}, U_{R_0}) \cup \mathbf{J}(\mathbf{x}^{(2)}, U_{R_0}), \forall h \in]0, T], |x_i^{(1)} - x_i^{(2)}| \leqslant \varepsilon_0 \text{ and } \Delta_{0,T,h}(w_i) \leqslant 2h^{\kappa_1} \\
&(3.13) \qquad \xi^{(1)}(\cdot), \xi^{(2)}(\cdot) \in \Lambda(\varepsilon_1, \varepsilon_2, \delta, T, M, R_0).
\end{aligned}$$

If $2(\varepsilon_0 + \varepsilon_1) < \varepsilon_2 < \frac{r}{M}$ and k satisfies $k(Mr + R) \leqslant R_0 \leqslant R_1$, then, there exists a constant C_7 such that for all indices a satisfying $|x_a^{(1)}| \leqslant R_0 - k(Mr + R)$ and for all $t \in [0, \delta]$,

$$|\xi_a^{(1)}(t) - \xi_a^{(2)}(t)| \leqslant C_7 e^{KC_7\delta} \varepsilon_0 + \frac{(KC_7\delta)^k}{k!}(\varepsilon_0 + 2\varepsilon_1) + 3C_7\delta g(R) e^{KC_7\delta}.$$

(Recall that K is the Lipschitz constant of $\nabla\Phi_s$ defined in (1.7)).

Proof. Put $R' = Mr + R$. By (3.13) for any $a \in \mathbf{J}(\mathbf{x}^{(\alpha)}, U_{R_0 - R'})$,

$$\Delta_{0,T,\delta}(\xi_a^{(\alpha)}(\cdot)) \leqslant \varepsilon_1, \quad \sharp \mathbf{C}(r + \varepsilon_2, x_a^{(\alpha)}, \mathbf{x}^{(\alpha)}) \leqslant M,\ \alpha = 1, 2.$$

Since $\varepsilon_2 < r/M$, we see that

$$\mathbf{C}(r + \varepsilon_2, x_a^{(\alpha)}, \mathbf{x}^{(\alpha)}) \subset U_{R_0 - R - \varepsilon_2},\ \alpha = 1, 2.$$

We put

$$\begin{aligned}
\mathbb{J}(a) &= \mathbf{J}(\mathbf{x}^{(1)}, \mathbf{C}(r + \varepsilon_2, x_a^{(1)}, \mathbf{x}^{(1)})), \\
\mathbb{J}_R(a) &= \mathbf{J}(\mathbf{x}^{(1)}, U_{R+\varepsilon_2}(\mathbf{C}(r + \varepsilon_2, x_a^{(1)}, \mathbf{x}^{(1)}))).
\end{aligned}$$

Then, since $(M - 1)(r + \varepsilon_2) + R + \varepsilon_2 \leqslant R'$, we have

$$(3.14) \qquad a \in \mathbb{J}(a) \subset \mathbb{J}_R(a) \subset \mathbf{J}(\mathbf{x}^{(1)}, U_{R'}(x_a^{(1)})) \text{ and } \sharp\mathbb{J}(a) \leqslant M.$$

By (3.12) and (3.13), for $t \in [0, \delta]$

$$\begin{aligned}
&|\xi_i^{(1)}(t) - \xi_j^{(1)}(t)| \geqslant \varepsilon_2 - 2\varepsilon_1 > 0, \text{ if } i \in \mathbb{J}(a), j \notin \mathbb{J}(a), \\
&|\xi_i^{(2)}(t) - \xi_j^{(2)}(t)| \geqslant \varepsilon_2 - 2\varepsilon_1 - 2\varepsilon_0 > 0, \text{ if } i \in \mathbb{J}(a), j \notin \mathbb{J}(a).
\end{aligned}$$

Then we have, for $\alpha = 1, 2, t \in [0, \delta]$, and $i \in \mathbb{J}(a)$,

$$\xi_i^{(\alpha)}(t) = x_i^{(\alpha)} + W_i^{(\alpha)}(t) + \sum_{j \in \mathbb{J}(a)} \int_0^t (\xi_i^{(\alpha)}(s) - \xi_j^{(\alpha)}(s)) d\rho_{ij}^{(\alpha)}(s),$$

where

$$W_i^{(\alpha)}(t) = w_i(t) - \frac{1}{2} \sum_{j \in \mathbf{J}(\alpha)} \int_0^t \nabla\Phi_s(\xi_i^{(\alpha)}(s) - \xi_j^{(\alpha)}(s)) ds + \int_0^t c_i^{(\alpha)}(s, \xi^{(\alpha)}(s)) ds.$$

For $\mathbf{x} = (x_1, x_2, \dots)$ such that $\mathbf{x} = \{x_1, x_2, \dots\} \in \mathfrak{X}$ and for any nonempty finite subset $\mathbf{J}$ of $\mathbb{N}$, we denote by $\mathbf{x_J} = (x_i, i \in \mathbf{J}) \in (\mathbb{R}^d)^{\mathbf{J}}$ the projection of $\mathbf{x}$ on $(\mathbb{R}^d)^{\mathbf{J}}$ and by $|\mathbf{x}|_{\mathbf{J}} = \max_{i \in \mathbf{J}} |x_i|$ its supremum norm.

Since each process $\xi_{\mathbb{J}(a)}^{(\alpha)}(.)$ is solution of a Skorohod equation, we can apply Lemma 2.2 for a state space dimension $m = \sharp\mathbb{J}(a)d$ bounded by Md and for $w = W_{\mathbb{J}(a)}^{(\alpha)}$.

Remark that

$$\forall h \in]0, T], \quad \Delta_{0,T,h}(W_i^{(\alpha)}) \leqslant 2h^{-\kappa_1} + \overline{\nabla\Phi_s} h/2 + g(R)h,$$

which tends uniformly to 0 for $R > 0$ and $i \in \mathbb{J}(a)$ when h tends to 0. So by Lemma 2.3 there exists a constant $C > 0$ such that

$$\forall t \in [0, \delta], \forall i \in \mathbb{J}(a), \| \sum_{j \in \mathbb{J}(a)} \int_0^t (\xi_i^{(\alpha)}(s) - \xi_j^{(\alpha)}(s)) d\rho_{ij}^{(\alpha)}(s) \|_t \leqslant C.$$

This implies

$$\begin{aligned} |\xi_i^{(1)}(t) - \xi_i^{(2)}(t)| &\leqslant |\xi_{\mathbb{J}(a)}^{(1)}(t) - \xi_{\mathbb{J}(a)}^{(2)}(t)| \\ &\leqslant \exp\left(2\sqrt{M} C_1 C\right) \left(|\mathbf{x}_{\mathbb{J}(a)}^{(1)} - \mathbf{x}_{\mathbb{J}(a)}^{(2)}| + \|W_{\mathbb{J}(a)}^{(1)} - W_{\mathbb{J}(a)}^{(2)}\|_t\right) \\ &\leqslant \frac{C_7}{M} \left(|\mathbf{x}_{\mathbb{J}(a)}^{(1)} - \mathbf{x}_{\mathbb{J}(a)}^{(2)}| + \|W_{\mathbb{J}(a)}^{(1)} - W_{\mathbb{J}(a)}^{(2)}\|_t\right), \end{aligned}$$

where $C_7 = M \exp(2\sqrt{M} C_1 C)$.

By (3.12) and (3.13) we have $|\xi^{(1)}(t) - \xi^{(2)}(t)|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{kR'}(x_a^{(1)}))} \leqslant \varepsilon_0 + 2\varepsilon_1$. Thus assumption (1.8) holds, and together with assumption (1.7) on $\nabla\Phi_s$, we have

$$\begin{aligned} &\|W_{\mathbb{J}(a)}^{(1)} - W_{\mathbb{J}(a)}^{(2)}\|_t \\ &\leqslant \frac{1}{2} \sum_{i \in \mathbb{J}(a)} \sum_{j \in \mathbb{J}_R(a)} \int_0^t |\nabla\Phi_s(\xi_i^{(1)}(s) - \xi_j^{(1)}(s)) - \nabla\Phi_s(\xi_i^{(2)}(s) - \xi_j^{(2)}(s))| ds \\ &\quad + \frac{1}{2} \sum_{i \in \mathbb{J}(a)} \sum_{j \notin \mathbb{J}_R(a)} \int_0^t \left(|\nabla\Phi_s(\xi_i^{(1)}(s) - \xi_j^{(1)}(s))| + |\nabla\Phi_s(\xi_i^{(2)}(s) - \xi_j^{(2)}(s))|\right) ds \\ &\quad + \sum_{i \in \mathbb{J}(a)} \int_0^t \left(|c_i^{(1)}(\xi^{(1)}(s))| + |c_i^{(2)}(\xi^{(2)}(s))|\right) ds \\ &\leqslant \frac{1}{2} \sum_{i \in \mathbb{J}(a)} K \int_0^t \max_{j \in \mathbb{J}_R(a)} |\xi_i^{(1)}(s) - \xi_j^{(1)}(s) - \xi_i^{(2)}(s) + \xi_j^{(2)}(s)| ds + 3Mg(R)t \\ &\leqslant MK \int_0^t |\xi^{(1)}(s) - \xi^{(2)}(s))|_{\mathbb{J}_R(a)} ds + 3Mg(R)t. \end{aligned}$$

Then we have, for $t \in [0,\delta]$,

$$
\begin{aligned}
&|\xi_a^{(1)}(t) - \xi_a^{(2)}(t)| \\
&\leqslant C_7 |\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_{\mathbb{J}(a)} + KC_7 \int_0^t |\xi^{(1)}(s) - \xi^{(2)}(s)|_{\mathbb{J}_R(a)} ds + 3C_7 g(R) t.
\end{aligned} \tag{3.15}
$$

From (3.14) we see that if $a \in \mathbf{J}(\mathbf{x}^{(1)}, U_{R_0-2R'})$ and $i \in \mathbb{J}_R(a)$, then $i \in \mathbf{J}(\mathbf{x}^{(1)}, U_{R_0-R'})$ and so we can apply the above computation to the i^{th} coordinate : for $t \in [0,\delta]$,

$$
\begin{aligned}
&|\xi_i^{(1)}(t) - \xi_i^{(2)}(t)| \\
&\leqslant C_7 |\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_{\mathbb{J}(i)} + KC_7 \int_0^t |\xi^{(1)}(s) - \xi^{(2)}(s)|_{\mathbb{J}_R(i)} ds + 3C_7 g(R) t.
\end{aligned} \tag{3.16}
$$

Since $\mathbb{J}_R(i) \subset \mathbf{J}(\mathbf{x}^{(1)}, U_{2R'}(x_a^{(1)}))$, from (3.14), (3.15) we have for each $t \in [0,\delta]$

$$
\begin{aligned}
&|\xi_a^{(1)}(t) - \xi_a^{(2)}(t)| \\
&\leqslant C_7 |\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{2R'}(x_a^{(1)}))} \\
&+ KC_7 \int_0^t \Big(C_7 |\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{2R'}(x_a^{(1)}))} \\
&\qquad\qquad + KC_7 \int_0^s |\xi^{(1)}(u) - \xi^{(2)}(u)|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{2R'}(x_a^{(1)}))} du + 3C_7 g(R) s \Big) ds \\
&+ 3C_7 g(R) t \\
&\leqslant C_7 (1 + KC_7 t) |\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{2R'}(x_a^{(1)}))} \\
&+ (KC_7)^2 \int_0^t \int_0^s |\xi^{(1)}(u) - \xi^{(2)}(u)|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{2R'}(x_a^{(1)}))} \; du \; ds \\
&+ 3C_7 g(R)(t + KC_7 t^2/2).
\end{aligned}
$$

Repeating this procedure, we obtain for $a \in \mathbf{J}(\mathbf{x}^{(1)}, U_{R_0-kR'})$

$$
\begin{aligned}
&|\xi_a^{(1)}(t) - \xi_a^{(2)}(t)| \\
&\quad\leqslant C_7 \exp(KC_7 t) |\mathbf{x}^{(1)} - \mathbf{x}^{(2)}|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{kR'}(x_a^{(1)}))} \\
&\quad+ (KC_7)^k \int_0^t \int_0^{t_1} \cdots \int_0^{t_{k-1}} |\xi^{(1)}(t_k) - \xi^{(2)}(t_k)|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{kR'}(x_a^{(1)}))} \; dt_k \cdots dt_2 \; dt_1 \\
&\quad+ 3C_7 g(R) t \exp(KC_7 t).
\end{aligned}
$$

Once more, by (3.12) and (3.13) we have $|\xi^{(1)}(t) - \xi^{(2)}(t)|_{\mathbf{J}(\mathbf{x}^{(1)}, U_{kR'}(x_a^{(1)}))} \leqslant \varepsilon_0 + 2\varepsilon_1$. Then we obtain the desired estimate. ■

For $m \geqslant m_0$ large enough such that $4m^{-\kappa_1} < m^{-\kappa_2} < \frac{r}{M}$, and for ℓ in the following interval $m^p \leqslant \ell < (m+1)^p$, we put $\mathbf{J}(1) = \mathbf{J}(\mathbf{X}_0(\omega), U_\ell)$, $\mathbf{J}(2) = \mathbf{J}(\mathbf{X}_0(\omega), U_{\ell+1})$,

$$
\begin{aligned}
c_i^{(1)}(\mathbf{x}) &= -\frac{1}{2} \sum_{j, |X_j(\omega)| \geqslant \ell} \nabla \Phi_s(x_i - X_j(\omega)) - \frac{1}{2} \nabla \psi^{\ell, \mathbf{X}_0(\omega)}(x_i), \\
c_i^{(2)}(\mathbf{x}) &= -\frac{1}{2} \sum_{j, |X_j(\omega)| \geqslant \ell+1} \nabla \Phi_s(x_i - X_j(\omega)) - \frac{1}{2} \nabla \psi^{\ell+1, \mathbf{X}_0(\omega)}(x_i),
\end{aligned}
$$

$\varepsilon_1 = m^{-\kappa_1}$, $\varepsilon_2 = m^{-\kappa_2}$, $\delta = \frac{1}{m}$, $R = m^{p-3}$, $R_1 = \ell - R$.

For $t \in [0, \frac{1}{m}]$ we put $\mathbf{x}^{(1)} = \mathbf{x}^{(2)} = \mathbf{X}_0(\omega)$, $\varepsilon_0 = 0$, $R_0 = 2Tm^{p-1}$ and apply Lemma 3.3 with $k = m$. Then, there exists m_1 such that, for $m \geqslant m_1$,

$$|X_i^\ell(\omega,t) - X_i^{\ell+1}(\omega,t)| \leqslant a(m),\ i \in \mathbf{J}(\mathbf{X}_0(\omega), U_{(2Tm^2-m)R}),\ t \in [0, \frac{1}{m}],$$

where

$$a(m) = \frac{2}{m!}(\frac{KC_7}{m})^m m^{-\kappa_1} + \frac{3C_7}{m} g(m^{p-3}) \exp(KC_7/m).$$

For $t \in [\frac{1}{m}, \frac{2}{m}]$ we put $\mathbf{x}^{(1)} = \mathbf{X}^\ell(\omega, \frac{1}{m})$, $\mathbf{x}^{(2)} = \mathbf{X}^{\ell+1}(\omega, \frac{1}{m})$, $\varepsilon_0 = a(m)$, $R_0 = (2Tm^2 - m)R$ and apply Lemma 3.3 with $k = m$. Then, we have

$$|X_i^\ell(\omega,t) - X_i^{\ell+1}(\omega,t)| \leqslant (1 + C_9) a(m),\ i \in \mathbf{J}(\mathbf{X}_0(\omega), U_{(2Tm^2-2m)R}),\ t \in [\frac{1}{m}, \frac{2}{m}],$$

where $C_9 = \sup_{m > m_1} (C_7 \exp(\frac{KC_7}{m}) + \frac{1}{m!}(\frac{KC_7}{m})^m) < +\infty$.

Repeating this procedure, we have for all $i \in \mathbf{J}(\mathbf{X}_0(\omega), U_{Tm^{p-1}})$, $\ell \in [m^p, (m+1)^p]$ and $t \in [0, T]$,

$$|X_i^\ell(\omega,t) - X_i^{\ell+1}(\omega,t)| \leqslant a(m) \sum_{k=0}^{mT-1} C_9^k = a(m) \frac{C_9^{mT} - 1}{C_9 - 1}.$$

Hence, we can choose $m_2 \geqslant m_1$ such that for any $m \geqslant m_2$ and $i \in \mathbf{J}(\mathbf{X}_0(\omega), U_{Tm^{p-1}})$,

$$\begin{aligned}
\sum_{\ell=m^p}^{\infty} \sup_{t\in[0,T]} |X_i^\ell(\omega,t) - X_i^{\ell+1}(\omega,t)| &\leqslant \sum_{n=m}^{\infty} \sum_{\ell=n^p}^{(n+1)^p} a(n) \frac{C_9^{nT} - 1}{C_9 - 1} \\
&\leqslant \sum_{n=m}^{\infty} (n+1)^p a(n) \frac{C_9^{nT}}{C_9 - 1} \\
&\leqslant C_{10} \sum_{n=m}^{\infty} \big(\frac{1}{n!} + C_9^{nT} \exp(-\beta_1 n^{\beta_2(p-3)})\big)
\end{aligned}$$

This series converges if we choose the parameter $p \geqslant p_0 > 1/\beta_2 + 3$ (we recall that β_2 is the exponent appearing in the exponential decreasing of $\nabla\Phi_s$ (1.8)).

So there exists $\mathbf{X}^\infty(\cdot)$ such that, for all $i \in \mathbf{J}(\mathbf{X}(\omega), U_{Tm^{p-1}})$ and $\ell > m^p$,

$$\sup_{t\in[0,T]} |X_i^\ell(\omega,t) - X_i^\infty(\omega,t)| < \frac{C_{11}}{m!}. \tag{3.17}$$

Thus we obtain

$$P(\lim_{\ell\to\infty} \sup_{t\in[0,T]} |X_i^\ell(t) - X_i^\infty(t)| = 0,\ T > 0,\ i \in \mathbb{N}) = 1,$$

which is exactly the condition for $\mathbf{X}^\ell$ to converge a.s. in $W(\mathfrak{X})$ to $\mathbf{X}^\infty$.

Since the process $\mathbf{X}^\infty(\cdot)$ is the limit of $\mathbf{X}^\ell(\cdot)$, the reversibility of $\mathbf{X}^\infty(\cdot)$ is a consequence of the reversibility property for $\mathbf{X}^\ell(\cdot)$ (see the proof of Theorem 2 in [Tan96]).

Remark that any canonical Gibbs state associated to the potential Φ is also a reversible state for the process $\mathbf{X}^\infty$, since it is a mixture of Gibbs states (with respect to the activity parameter z).

From Lemma 3.2 and (3.17) we easily have

$$\sum_{m=1}^{\infty} P(\mathbf{X}^\infty(\cdot) \in \Lambda(m^{-\kappa_1}, m^{-\kappa_2}, \frac{1}{m}, T, M, m^p)^c) < \infty,$$

and so $P(\mathbf{X}^\infty(\cdot) \in \mathcal{C} \cap \Theta) = 1$.

The proof of Proposition 3.1 will be complete by proving the following lemma.

Lemma 3.4 *The process $\mathbf{X}^\infty(t)$ is the unique solution of* (1.12) *with initial condition equal to $\mathbf{X}_0 = \{X_1, X_2, \dots\}$.*

Proof. By the same argument as in the proof of Lemma 3.3 for any $a \in \mathbb{N}$ and sufficiently large ℓ we have finite subsets $\mathbb{J}^\ell(\frac{k}{m}, a) \ni a$, $k = 0, 1, \dots, mT$ such that

$$|X_i^\ell(t) - X_j^\ell(t)| > m^{-\kappa_1}, \; i \in \mathbb{J}^\ell(\frac{k}{m}, a), j \notin \mathbb{J}^\ell(\frac{k}{m}, a), \; t \in [\frac{k}{m}, \frac{k+1}{m}].$$

By virtue of the estimate (3.17) for sufficiently large ℓ we can chose $\mathbb{J}^\ell(\frac{k}{m}, a)$, $k = 0, 1, \dots, mT$ to be independent of ℓ and denote them by $\mathbb{J}(\frac{k}{m}, a)$, $k = 0, 1, \dots, mT$. Then we have

$$\begin{aligned} X_i^\ell(t) &= X_i^\ell(\frac{k}{m}) + B_i(t) - B_i(\frac{k}{m}) - \frac{1}{2}\sum_j \int_{\frac{k}{m}}^t \nabla\Phi_s(X_i^\ell(s) - X_j^\ell(s))ds \\ &+ \sum_{j \in \mathbb{J}(\frac{k}{m}, a)} \int_{\frac{k}{m}}^t (X_i^\ell(s) - X_j^\ell(s)) dL_{ij}^\ell(s), \; i \in \mathbb{J}(\frac{k}{m}, a), \; t \in [\frac{k}{m}, \frac{k+1}{m}]. \end{aligned}$$

By Lemma 2.2 we obtain

$$\begin{aligned} X_i^\infty(t) &= X_i^\infty(\frac{k}{m}) + B_i(t) - B_i(\frac{k}{m}) - \frac{1}{2}\sum_j \int_{\frac{k}{m}}^t \nabla\Phi_s(X_i^\infty(s) - X_j^\infty(s))ds \\ &+ \sum_{j \in \mathbb{J}(\frac{k}{m}, a)} \int_{\frac{k}{m}}^t (X_i^\infty(s) - X_j^\infty(s)) dL_{ij}^\infty(s), \; i \in \mathbb{J}(\frac{k}{m}, a), \; t \in [\frac{k}{m}, \frac{k+1}{m}], \end{aligned} \tag{3.18}$$

which implies that $\mathbf{X}^\infty(\cdot)$ is a solution of (1.12).

Suppose that $\mathbf{Y}(\cdot)$ is also a solution of (1.12). Let $\mathbf{X}^\infty(\cdot, \omega) \in \Theta$ and $\mathbf{Y}(\cdot, \omega) \in \mathcal{C}$. For any $T \in \mathbb{N}$, $m_3 \in \mathbb{N}$ and $p \geqslant p_0$, we can choose $0 < \kappa < \kappa' < \frac{1}{2}$ and $m \geqslant m_3$ with $m^{-\kappa} > 4m^{-\kappa'}$ such that

$$\mathbf{Y}(\cdot, \omega) \in \mathcal{C}[m^{-\kappa}, \frac{1}{m}, T, M, m^p], \quad \mathbf{X}^\infty(\cdot, \omega) \in \Theta[m^{-\kappa'}, \frac{1}{m}, T, m^p].$$

Then we can take a sequence $\mathbb{J}(\frac{k}{m},a) \ni a$ for which (3.18) holds for each $k = 0,1,\dots,mT$ and

$$\begin{aligned} Y_i(t) &= Y_i(\frac{k}{m}) + B_i(t) - B_i(\frac{k}{m}) - \frac{1}{2}\sum_j \int_{\frac{k}{m}}^t \nabla\Phi_s(Y_i(s)-Y_j(s))ds \\ &+ \sum_{j\in\mathbb{J}(\frac{k}{m},a)} \int_{\frac{k}{m}}^t (Y_i(s)-Y_j(s))dL_{ij}^Y(s),\ i\in\mathbb{J}(\frac{k}{m},a),\ t\in[\frac{k}{m},\frac{k+1}{m}]. \end{aligned}$$

Then by the same argument to get (3.17) we have

$$\sup_{t\in[0,T]} |Y_i(\omega,t) - X_i^\infty(\omega,t)| < \frac{C_{12}}{m!},\ i\in\mathbf{J}(\mathbf{X}_0(\omega),U_{Tm^{p-1}}).$$

Since we can take m as large as we want, we have $\mathbf{X}(t,\omega) = \mathbf{Y}(t,\omega), t\in[0,T]$, for any $T>0$ and ω in a set of full probability. This completes the proof of Proposition 3.1. ■

4 Solution with deterministic initial condition and measurability properties

For $\mathbf{x} = \{x_1,x_2,\dots\} \in \mathfrak{X}$ and $\mathbf{w} = (w_1,w_2,\dots) \in W_0(\mathbb{R}^d)^{\mathbb{N}}$, we consider the following system of equations (4.1) under the conditions (4.2) and (4.3):

(4.1) $\forall i\in\mathbb{N}$,
$$\xi_i(t) = x_i + w_i(t) - \frac{1}{2}\sum_{j\in\mathbb{N}}\int_0^t \nabla\Phi_s(\xi_i(s)-\xi_j(s))ds + \sum_{j\in\mathbb{N}}\int_0^t (\xi_i(s)-\xi_j(s))d\rho_{ij}(s)$$

(4.2) $\xi(\cdot) = \{\xi_1(\cdot),\xi_2(\cdot),\cdots\}_i \in \mathcal{C}\cap\Theta$

(4.3) $\rho_{ij}, i,j\in\mathbb{N}$ are continuous nondecreasing functions with $\rho_{ij}(0)=0$, $\rho_{ij}=\rho_{ji}$ and

$$\rho_{ij}(t) = \int_0^t \mathbb{1}_{\{r\}}(|\xi_i(s)-\xi_j(s)|)d\rho_{ij}(s).$$

We denote by Ξ the set of all elements $(\mathbf{x},\mathbf{w})$ of $\mathfrak{X}\times W_0(\mathbb{R}^d)^{\mathbb{N}}$ for each of which there exists a solution $\xi(t,\mathbf{x},\mathbf{w})$ of (4.1). By the same argument as in the proof of Lemma 3.4, we see that $\xi(t,\mathbf{x},\mathbf{w})$ is the unique solution of (4.1). Remark that if $\mathbf{x}(\cdot)\in\mathcal{C}\cap\Theta$, then $\mathbf{x}(s+\cdot)\in\mathcal{C}\cap\Theta$. So, for $(\mathbf{x},\mathbf{w})\in\Xi$,

(4.4) $(\xi(s,\mathbf{x},\mathbf{w}),\theta_s\mathbf{w}) \in \Xi, \quad s\geqslant 0,$

and

(4.5) $\xi(s+t,\mathbf{x},\mathbf{w}) = \xi(t,\xi(s,\mathbf{x},\mathbf{w}),\theta_s\mathbf{w}), \quad s,t\geqslant 0,$

by virtue of the uniqueness, where $\theta_s \mathbf{w}(t) = \mathbf{w}(s+t) - \mathbf{w}(s)$. Put

$$\widehat{\xi}(t,\mathbf{x},\mathbf{w}) = \begin{cases} \xi(t,\mathbf{x},\mathbf{w}) & \text{if } (\mathbf{x},\mathbf{w}) \in \Xi \\ \mathbf{x} & \text{otherwise ,} \end{cases}$$

for $t \geqslant 0$. Similarly as in lemma 6.1 in [Tan96], we can prove that :
(4.6) Ξ is $\mathcal{B}(\mathfrak{X} \times W_0(\mathbb{R}^d)^{\mathbb{N}})-$measurable
(4.7) $(\cdot, \mathbf{x}, \mathbf{w}) \longmapsto \widehat{\xi}(\cdot, \mathbf{x}, \mathbf{w})$ is measurable from $\mathfrak{X} \times W_0(\mathbb{R}^d)^{\mathbb{N}}$ to $W(\mathbb{R}^d)^{\mathbb{N}}$ endowed with their Borel fields.

End of the proof of Theorem 1.4 (i).
By Fubini's theorem

$$\mathfrak{Y} = \{\mathbf{x} \in \mathfrak{X} : P_W^{\otimes \mathbb{N}}(\Xi_{\mathbf{x}}) = 1\}$$

is a measurable subset of $\mathfrak{X}$ where $\Xi_{\mathbf{x}} = \{\mathbf{w} \in W_0(\mathbb{R}^d)^{\mathbb{N}} : (\mathbf{x},\mathbf{w}) \in \Xi\}$. By Proposition 3.1, if the distribution of $\mathbf{X}$ is $\mu \in \mathcal{G}(z,\Phi)$, for some $z > 0$, then $P((\mathbf{X},\mathbf{B}) \in \Xi) = 1$, and so $P(\mathbf{X} \in \mathfrak{Y}) = 1$. We put

$$P(t,\mathbf{x},\Lambda) = P_W^{\otimes \mathbb{N}}(\xi(t,\mathbf{x},\cdot) \in \Lambda),$$

for $t \geqslant 0$, $\mathbf{x} \in \mathfrak{Y}$ and $\Lambda \in \mathcal{B}(\mathfrak{Y})$. Suppose that $\mathbf{x} \in \mathfrak{Y}$. Then $(\mathbf{x},\mathbf{B}) \in \Xi$, a.s. and so by (4.4) and (4.5)

$$(\xi(\tau,\mathbf{x},\mathbf{B}), \theta_\tau \mathbf{B}) \in \Xi, \quad \text{a.s. and } \xi(\tau + t,\mathbf{x},\mathbf{B}) = \xi(t, \xi(\tau,\mathbf{x},\mathbf{B}), \theta_\tau \mathbf{B}), \quad \text{a.s.}$$

for any $\mathcal{F}_t-$stopping time τ. From the strong Markov property of $\mathbf{B}$ we see that $\xi(\tau,\mathbf{x},\mathbf{B}) \in \mathfrak{Y}$ a.s. and

$$\begin{aligned} P(\xi(\tau+t,\mathbf{x},\mathbf{B}) \in \Lambda | \mathcal{F}_\tau) &= P_W^{\otimes \mathbb{N}}(\xi(t, \xi(\tau,\mathbf{x},\mathbf{B}), \cdot) \in \Lambda) \\ &= P(t, \xi(\tau,\mathbf{x},\mathbf{B}), \Lambda), \quad \text{a.s.} \end{aligned}$$

for $t \geqslant 0$ and $\Lambda \in \mathcal{B}(\mathfrak{Y})$. This means that $\xi(t,\mathbf{x},\mathbf{w})$ is a strong Markov process. ■

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