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Long time behavior of stochastic hard ball systems

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AbstractWe study the long time behavior of a system of $n = 2, 3$ Brownian hard balls, living in $\mathbb{R}^d$ for $d \geq 2$, submitted to a mutual attraction and to elastic collisions.

Keywords: Stochastic Differential Equations, hard core interaction, reversible measure, normal reflection, local time, Lyapunov function, Poincaré inequality.

1. Introduction and main results.

Consider n hard balls with radius $r/2$ and centers $X_1, ..., X_n$ located in $\mathbb{R}^d$ for some $d \geq 2$. They are moving randomly and when they meet, they are performing elastic collisions. We are interested in the long time behavior of such a dynamics, where the centers of the balls are moving according to a Brownian motion in a Gaussian type pair potential. It is modelised by the following system of stochastic differential equations with reflection

$$
\text{(A)} \begin{cases} \text{for } i \in \{1, \cdots, n\}, \, t \in \mathbb{R}^+, \\ X_i(t) = X_i(0) + W_i(t) - \, a \displaystyle\sum_{j=1}^{n} \int_0^t (X_i(s) - X_j(s)) ds + \sum_{j=1}^{n} \int_0^t (X_i(s) - X_j(s)) dL_{ij}(s) \,, \\ L_{ij}(0) = 0, \quad L_{ij} \equiv L_{ji} \quad \text{ and } \quad L_{ij}(t) = \displaystyle\int_0^t \mathbb{1}_{|X_i(s) - X_j(s)| = r} \, dL_{ij}(s), \quad L_{ii} \equiv 0, \end{cases}
$$

where $W_1, ..., W_n$ are n independent standard Wiener processes. The local time L_{ij} describes the elastic collision (normal mutual reflection) between balls i and j. The parameter a is assumed to be non-negative. Therefore the drift term derives from an attractive quadratic potential.

Note that the Markov process X satisfying (A) admits a unique (up to a multiplicative constant) unbounded invariant measure μ_a defined on $(\mathbb{R}^d)^n$ by:

$$
d\mu_a(\mathbf{x}) = e^{-a \sum_{i,j} |x_i - x_j|^2/2} \, \mathbb{1}_D(\mathbf{x}) \, d\mathbf{x}. \tag{1.1}
$$

Here $\mathbf{x} = (x_1, ..., x_n) \in (\mathbb{R}^d)^n$ and D is the interior of the set of allowed configurations i.e.

$$
D = \{\mathbf{x} \in (\mathbb{R}^d)^n \, ; \, |x_i - x_j| > r \text{ for all } i \neq j\} \,. \tag{1.2}
$$

Clearly the measure μ_a is invariant under the simultaneous translations of the n balls, that is under any transformation of the form $(x_1, ..., x_n) \mapsto (x_1 + u, ..., x_n + u)$, $u \in \mathbb{R}^d$.

Indeed we are even more interested by the intrinsic dynamics of the system, i.e. by the system of balls viewed from their center of mass, called $G := \frac{1}{n}(X_1 + ... + X_n)$. This (fictitious) point undergoes a Brownian motion in $\mathbb{R}^d$ with covariance $\frac{1}{n}$ Id (notice the absence of reflection term). Choosing G as the (moving) origin of the ambient space $\mathbb{R}^d$, we therefore consider the process Y of the relative positions, $Y_i = X_i - G, i = 1, \cdots, n$, which satisfies

$$
(\mathrm{B}) \begin{cases} \text{for } i \in \{1, \cdots, n\}, \, t \in \mathbb{R}^+, \\ Y_i(t) = Y_i(0) + \, M_i(t) - \, a \displaystyle\sum_{j=1}^{n} \int_0^t (Y_i(s) - Y_j(s)) ds + \sum_{j=1}^{n} \int_0^t (Y_i(s) - Y_j(s)) dL_{ij}(s) \\ L_{ij}(0) = 0, \quad L_{ij} \equiv L_{ji} \quad \text{and} \quad L_{ij}(t) = \displaystyle\int_0^t \mathbb{1}_{|Y_i(s) - Y_j(s)| = r} \, dL_{ij}(s), \quad L_{ii} \equiv 0. \end{cases}
$$

where the martingale term $(M_1, \cdots, M_n)$ is a new Brownian motion with covariation $\langle M_i, M_k \rangle(t) = \left(\frac{n-1}{n} \, \delta_{\{i=k\}} \, - \, \frac{1}{n} \, \delta_{\{i \neq k\}} \right) t \, \mathrm{Id}$.

The $(\mathbb{R}^d)^n$-valued Markov process $Y(t)$ admits as unique invariant probability measure

$$
d\pi_a(\mathbf{y}) = \, Z_a^{-1} \, e^{- \, a \, \sum_{i,j} |y_i - y_j|^2 / 2} \, \mathbb{1}_{D'}(\mathbf{y}) \, d\mathbf{y} \tag{1.3}
$$

for a well chosen normalization constant Z_a. The domain D', support of π_a obtained as linear transformation of D, is the following unbounded set

$$
D' := \{ \mathbf{y} \in (\mathbb{R}^d)^n \, ; \, |y_i - y_j| > r \; \text{ for all } i \neq j \, , \quad \sum_{i=1}^{n} y_i = 0 \} \, . \tag{1.4}
$$

Our aim is to describe the long time behavior of the process Y, that is of the system of balls viewed from their center of mass.

Before explaining in more details the contents of the paper, let us give an account of the existing literature and of related problems.

Existence and uniqueness of a strong solution for system (A) was first obtained in Y. Saisho and H. Tanaka (1986) with $a = 0$. Extensions to $a > 0$ and to $n = +\infty$ are done in M. Fradon and S. Roelly (2000, 2007); M. Fradon, S. Rœlly and H. Tanemura (2000). Random radii r were also studied in M. Fradon (2010); M. Fradon and S. Roelly (2010). The invariant (in fact reversible) measure for the system is discussed in Y. Saisho and H. Tanaka (1987) and M. Fradon and S. Roelly (2006) for an infinite number of balls.

The construction of the stationary process (i.e. starting from the invariant measure) can also be performed by using Dirichlet forms theory. Actually, D intersected with any ball $B(0, R) \subset (\mathbb{R}^d)^n$ is a Lipschitz domain (see the Appendix) so that one can use results in R. F. Bass and P. Hsu (1990); Z. Q. Chen, P. J. Fitzsimmons and R. J. Williams (1993); M. Fukushima and M. Tomisaki (1996) to build the Hunt process naturally associated to the Dirichlet form (see e.g. M. Fukushima, Y. Oshima and M. Takeda (1994) for the theory of Dirichlet forms)

$$
\mathcal{E}_a^R(f) = \int_{D \cap B(0,R)} |\nabla f|^2 \, d\mu_a \, . \tag{1.5}
$$

It is then enough to let R go to infinity and show conservativeness of the obtained process which is equivalent to non explosion. This is standard.

The solution of (A) built by using stochastic calculus do coincide with the Hunt process associated to the Dirichlet form $\mathcal{E}_a$ obtained for $R = +\infty$. Some properties, like the decomposition of the boundary

into a non-polar and a polar parts or the Girsanov's like structure are discussed in Z. Q. Chen, P. J. Fitzsimmons, M. Takeda, J. Ying and T.-S. Zhang (2004).
For an infinite number of balls, such a construction is performed in H. Osada (1996); H. Tanemura (1996, 1997).

Let us recall also some regularity of the processes and their associated semi-groups which we will need in the sequel. For $x \in \bar{D}$ we denote by $P_t(x, dy)$ the transition kernel of the process $X(.)$ starting from x at time t. It is well known that for all $x \in D$, $P_t(x, dy)$ is absolutely continuous with respect to the Lebesgue measure restricted to $\bar{D}$ (in particular does not charge the boundary ∂D). In addition the density $p_t(x, y)$ is smooth as a function of the two variables x and y in $D \times D$. This follows from standard elliptic estimates (as in R. F. Bass and P. Hsu (1991)) or from the use of Malliavin calculus as explained in P. Cattiaux (1986, 1992). Furthermore this density kernel extends smoothly up to the smooth part of the boundary (see P. Cattiaux (1987, 1992)). But since the domain is (locally) Lipschitz, the potential theoretic tools of R. F. Bass and P. Hsu (1991) sections 3 and 4 can be used to show that $(t, x, y) \mapsto p_t(x, y)$ extends continuously to $\mathbb{R}^+ \times \bar{D} \times \bar{D}$. Actually section 4 in R. F. Bass and P. Hsu (1991) is written for bounded Lipschitz domain but extends easily to our situation by localizing the Dirichlet form as we mentioned earlier and using conservativeness (of course the function is no more uniformly continuous). In particular the process is Feller (actually strong Feller thanks to M. Fukushima and M. Tomisaki (1996)).
Comparison with the killed process at the boundary shows that for any $t > 0$ and any starting $x \in D$, $p_t(x, y) > 0$ for any $y \in D$ (see e.g. (3.15) and (3.16) in R. F. Bass and P. Hsu (1991) and use repeatedly the Chapman-Kolmogorov relation to extend the result to all t and y introducing a chaining from x to y). The previous continuity thus implies that for all $t > 0$, all compact subsets K and K' of $\bar{D}$ there exists a constant $C(t, K, K') > 0$ such that

$$\text{for all } x \in K \text{ and } y \in K', \quad p_t(x, y) \geq C(t, K, K') \,. \tag{1.6}$$

In particular compact sets are "petite sets" in the Meyn-Tweedie terminology S. P. Meyn and R. L. Tweedie (1993) and for any compact set $K \subset \bar{D}$ and any $t > 0$, the

$$(\textit{Local Dobrushin condition}) \qquad \sup_{x, x' \in K} \parallel P_t(x, dy) - P_t(x', dy) \parallel_{TV} < 2 \,, \tag{1.7}$$

is fulfilled, where $\parallel . \parallel_{TV}$ denotes the total variation distance.

Another classical consequence is the uniqueness of the invariant measure (since all invariant measures are actually equivalent) up to a multiplicative constant.

Since Y is deduced from X by a smooth linear transformation, similar statements are available for Y in D'. In particular Y is a Feller process satisfying the local Dobrushin condition with a unique invariant probability measure.

Looking at long time behavior of such systems is not only interesting by itself but relies, as $a \to +\infty$ (low temperature regime in statistical mechanics), to the following *finite packing problem*: what is the shape of a cluster of n spheres - with equal radii $r/2$ - minimizing their quadratic energy, i.e. their second moment about their center of mass. (For a review of different questions on finite packing, see the recent monograph K. Böröczky (2004)). This problem, in spite of its simple statement and its numerous useful applications, remains mainly open. Even for $d = 2$ (so called *penny-packings*), only the case $n \leq 7$ was solved by Temesvari in A. H. Temesvári (1974). For more pennies, the optimal configurations are known only among the specific class of hexagonal packings T. Y. Chow (1995). For $d = 3$ one finds in N.J.A. Sloane, R.H. Hardin, T.D.S. Duff and J.H. Conway (1995) a description of

the putatively optimal arrangements until $n \leq 32$. For the case of infinitely many spheres and their celebrated densest packing, we refer to J.H. Conway and N.J.A. Sloane (1993) or to M. Fradon and S. Roelly (2006) p.99-100 for recent references and a more complete discussion.
Indeed, as $a \to +\infty$, the invariant measure π_a concentrates on the set of configurations with minimal quadratic energy i.e. the set

$$\mathbf{E}_{min} = \{\mathbf{y} \in D' \,;\, V(\mathbf{y}) := \sum_{i,j} |y_i - y_j|^2 = \inf_{\mathbf{z} \in \mathcal{D}'} \sum_{i,j} |z_i - z_j|^2\} ,$$

which obviously depends on n, r and d. So looking simultaneously at large t and large a furnishes some *simulated annealing algorithm* for the uniform measure on $\mathbf{E}_{min}$ (see Theorem 3.1 for the case $n = 3$).

A similar (but different) algorithmic point of view is discussed in the recent paper P. Diaconis, G. Lebeau and L. Michel (2011), where the problem under discussion is: *how can we place* randomly n *hard balls of radius r in a given large ball (or hypercube)?* According to the introduction of P. Diaconis, G. Lebeau and L. Michel (2011) this problem is the origin of Metropolis algorithm. The authors rely the asymptotics of the spectral gap of a discrete Metropolis algorithm to the first Neumann eigenvalue (called ν_1) for the Laplace operator in D' intersected with a large hypercube (see Theorem 4.6).

There are several methods to attack the study of long time behavior for Markov processes. In this paper we will restrict ourselves to exponential (or geometric) ergodicity. Moreover we will try to give some controls on the rate of exponential ergodicity. Let us first recall some definitions.

Definition 1.8. *A Markov process Z with transition distribution P_t and invariant measure π is said to be exponentially ergodic if there exists $\beta > 0$ such that for all initial condition z,*

$$\| \, P_t(z,.) - \pi \, \|_{TV} \leq \, C(z) \, e^{-\beta \, t} \, .$$

If the function $z \mapsto C(z)$ is μ-integrable, the previous extends to any initial distribution μ.
Our main result reads as follows.

Theorem 1.9. *Consider a system of n hard balls in $\mathbb{R}^d$ submitted to the dynamics described by* (A). *If $n = 2, 3$, the process Y of their relative positions viewed from their center of mass, described by the system* (B), *is exponentially ergodic.*

Remark that, if we are only interested in the convergence of the ball system to the set of configurations with minimal energy in the large attraction regime, the quantity of interest reduces to the $(\mathbb{R}^+)^n$-valued system of the distances between the centers of the n balls and their center of mass. Its rate of convergence to equilibrium is much faster that those of the $(\mathbb{R}^d)^n$-valued process Y. For two balls, the difference is explicit when comparing Theorems 2.1 and 2.4 in the next section.

Exponential ergodicity is connected to the existence of an exponential coupling, as explained in A. M. Kulik (2011, 2009), and is strongly dependent on the existence of exponential moments for the hitting time of compact subsets. This method can be traced back to A.Yu. Veretennikov (1987). Let us give a precise statement taken from A. M. Kulik (2011) Theorem 2.2.

Theorem 1.10. *Suppose that the process Z satisfies the local Dobrushin condition. Assume that we can find a real valued function Φ and a compact set K and positive constants c, α such that*

1. *Φ is larger than 1 and $\Phi(z) \to +\infty$ as $|z| \to +\infty$,*
2. *there exists $\alpha > 0$ and $c > 0$ such that for all initial condition z,*

$$\mathsf{E}_z(\Phi(Z(t))\, \mathbb{1}_{\tau_K > t}) \leq c\, e^{-\alpha\, t}\, \Phi(z)$$

where τ_K denotes the hitting time of K,

3. *$\sup_{z \in K\,,\, t>0} \mathsf{E}_z(\Phi(Z(t))\, \mathbb{1}_{\Phi(Z(t))>M}) \to 0$ as $M \to +\infty$.*

Then the process Z is exponentially ergodic.

Though the result is only stated in the case $c = 1$ in A. M. Kulik (2011), the method extends to any $c > 0$ without difficulty. It is instructive to compare this result with other forms of *Harris-type* theorems for exponential ergodicity, frequently used in the literature. Typically such theorems assume an irreducibility of the process on a set K (in our case this is the local Dobrushin condition) and its recurrence. Recurrence assumption is formulated usually in the terms of the generator $\mathcal{L}$ of the process, like the Foster-Lyapunov condition, see e.g. S. P. Meyn and R. L. Tweedie (1993),

$$\mathcal{L}\Phi \leq -\alpha\, \Phi \,+\, C\, \mathbb{1}_K. \tag{1.11}$$

In Theorem 1.10, assumptions (2) and (3) can be interpreted as a recurrence assumption, but since the generator $\mathcal{L}$ is not involved therein, we call it an *integral* Lyapunov condition, while (1.11) is a *differential* one. In our framework, because of the presence of several local time terms in (B), it is very difficult to find a Lyapunov function which satisfy (1.11) but we succeeded in showing that the quadratic energy of the system, V, satisfies the more tractable integral Lyapunov condition presented in Theorem 1.10.

It has also to be noticed that, unfortunately, the exponential rate of convergence β in Theorem 1.10 is very difficult to express explicitly, as it depends on α but also on many other constants connected with the behavior of the process, in particular a quantitative version of the local Dobrushin condition in K.

Another classical approach of exponential ergodicity is the spectral approach, i.e. the existence of a *spectral gap*. Recall the well known equivalence

Proposition 1.12. *Pick $\theta > 0$. For any $f \in \mathbb{L}^2(\pi)$,*

$$Var_\pi(P_t f) \leq e^{-\theta t}\, Var_\pi(f)$$

if and only if π satisfies the following Poincaré inequality

$$Var_\pi(f) \leq \frac{1}{\theta} \int |\nabla f|^2\, d\pi\,.$$

When π is not only invariant but reversible, it is known that both approaches coincide, i.e.

Theorem 1.13. *The process Z is exponentially ergodic if and only if π satisfies some Poincaré inequality. Furthermore if π satisfies a Poincaré inequality, we may choose $\beta = \theta/2$, while if the process is exponentially ergodic we may choose $\theta = \beta$.*

The difficult part of this equivalence (i.e exponential ergodicity implies Poincaré) is shown in D. Bakry, P. Cattiaux, and A. Guillin (2008) Theorem 2.1 or A. M. Kulik (2011) Theorem 3.4. The converse direction is explained in P. Cattiaux, A. Guillin and P.A. Zitt (2013).

As it was pointed out in details in the recents A. M. Kulik (2011); P. Cattiaux, A. Guillin and P.A. Zitt (2013) (which despite the dates of publication were achieved simultaneously), these two formulations of exponential ergodicity are also equivalent to the existence of exponential moments for the hitting time of a compact set, in the case of usual diffusion processes. For the reflected processes we are looking at, some extra work is necessary.

In this paper, we use the second approach related to Theorem 1.13 to analyse the 2-ball system in the next section, and the first method presented in Theorem 1.10 to prove the ergodicity of the 3-ball system developped in the third section. For that system we prove geometric ergodicity in Theorem 3.1, but during the proof (see e.g. the statement of Proposition 3.4) we show that the total quadratic energy V is a Lyapunov function in the sense of the function Φ of Theorem 1.10. Hence we have an almost explicit exponential rate depending linearly of the attraction coefficient a at least for large a's.
The proof is merely intricate. The key idea is to study and control the hitting time of a "cluster" i.e. a set of relatively small quadratic energy. It turns out that the most practical way to describe the triangle configuration built by the three centers is to look at the medians of this triangle. The reason is that one has to control a single local time term.
Throughout the proofs we have tried to trace the constants as precisely as possible, in particular to obtain the convergence rate as an explicit function of a.

2. The case of two balls.

In this section we consider the "baby model" case $n = 2$. The relative position of the two balls is described by the $\mathbb{R}^d$-valued process $Y := Y_1 = \frac{X_1 - X_2}{2}$ which satisfies

$$Y(t) = Y(0) + B_1(t) - 2a \int_0^t Y(s)\, ds + 2 \int_0^t Y(s) dL_1(s)\,,$$

where B_1 is a Brownian motion with covariance $(1/2)\, Id$ and L_1 is the local time of Y on the centered sphere with radius $r/2$, i.e. Y is simply an Ornstein-Uhlenbeck process outside the ball of radius $r/2$ and normally reflected on the boundary of this ball. In particular

$$\pi_a(dy) = Z_a^{-1}\, e^{-4a|y|^2}\, \mathbb{1}_{|y|>r/2}\, dy$$

is simply a centered Gaussian measure restricted to $D' = \mathbb{R}^d - B(0, r/2)$.
This measure is thus spherically symmetric and radially log-concave, so that one can use the method in S. G. Bobkov (2003) in order to evaluate the Poincaré constant. The following result is a direct consequence of E. Boissard, P. Cattiaux, A. Guillin and L. Miclo (2013)

Theorem 2.1. *π_a satisfies a Poincaré inequality with constant $C_P(\pi_a) = 1/\theta_a$ satisfying*

$$\frac{1}{2}\left(\frac{1}{8a} + \frac{r^2}{4d}\right) \le \max\left(\frac{1}{8a}\,,\, \frac{r^2}{4d}\right) \le C_P(\pi_a) \le \frac{1}{4a} + \frac{r^2}{4d}\,.$$

As we explained before, this result captures both the rate of convergence to a "well packed" configuration and the rate of stabilization of an uniform rotation. If we want to avoid the last property we are led to look at the $\mathbb{R}^+$-valued process $y(t) := |Y(t)|$ which is the *radial Ornstein Uhlenbeck* process reflected at $r/2$ i.e. solves the following one dimensional (at $r/2$) reflected S.D.E.

$$dy(t) = \frac{1}{\sqrt{2}}\, dW(t) - 2a\, y(t)\, dt + \frac{d-1}{4y(t)}\, dt + 2\, y(t) dL(t)\,, \tag{2.2}$$

with a standard Brownian motion W. Its one dimensional reversible probability measure is

$$\nu_a(d\rho) = Z_\rho^{-1}\,\rho^{d-1}\,e^{-4a\rho^2}\,\mathbb{1}_{\rho>r/2}\,d\rho\,, \tag{2.3}$$

for which we have the following result which furnishes a bound of the rate of "packing" of two balls.

Theorem 2.4. *Let $(P_t)_{t\geq 0}$ be the transition distribution of the half distance $(y(t))_{t\geq 0}$ between the centers of the two balls moving according to the dynamics* (A).

$$\forall y > \frac{r}{2} \qquad \|\, P_t(y,.) - \nu_a \,\|_{TV} \leq\, C(y)\, e^{-4a\,t}\,.$$

Proof. Theorem 2.4 holds as soon as ν_a satisfies a Poincaré inequality with constant $C_P(\nu_a)$ satisfying $C_P(\nu_a) \leq \frac{1}{8a}$. This latter result is a simple application of Bakry-Emery criterion on the interval $\rho \geq r/2$. Recall that Bakry-Emery criterion tells us that provided $V''(\rho) \geq A > 0$ for all $\rho \in \mathbb{R}$, then the (supposed to be finite) measure $e^{-V(\rho)}d\rho$ satisfies a Poincaré inequality with constant $1/A$. The measure $e^{-V(\rho)}\,\mathbb{1}_{\rho>r/2}\,d\rho$ can be approximated, as $N \to +\infty$, by $e^{-(V(\rho)+N((r/2)-\rho)_+^4)}d\rho$ which still satisfies the same lower bound for the second derivative, uniformly in N, showing that Bakry-Emery criterion extends to the case of an interval. Finally, it is immediately seen that ν_a satisfies Bakry-Emery criterion with $A = 8a$. □

Sketch of the proof of Theorem 2.1:. Actually ν_a is the radial part of π_a. In polar coordinates $(\rho, s) \in \mathbb{R}\times S^{d-1}$, π_a factorizes as $\nu_a \otimes ds$ where ds is the normalized uniform measure on the sphere S^{d-1}, which satisfies a Poincaré inequality with constant $1/d$. Bobkov's method exposed in S. G. Bobkov (2003) and detailed in E. Boissard, P. Cattiaux, A. Guillin and L. Miclo (2013) Proposition 2.1, allows us to deduce the upper bound of Theorem 2.1. For the lower bound it is enough to consider linear functions. □

Remark 2.5. *The spectral gap ($\theta/2$ in Proposition 1.12) of linear diffusion processes can be studied by solving some O.D.E. For instance in our case, if we consider the process $z(t) := y(t)^2 - \frac{r^2}{4}$, it is an* affine diffusion *reflected at* 0*, i.e. it solves the reflected S.D.E.*

$$dz(t) = \sqrt{2}\,\sqrt{z(t)+(r^2/4)}\,dB_t + 4a\left(\frac{d}{8a} - \frac{r^2}{4} - z(t)\right)\,dt + dL_z(t)\,,$$

where $L_z(.)$ is proportional to the local time of the process z at 0*. For such linear processes it is shown in V. Linetsky (2005) section 6.2, that the spectral gap is given by $\theta/2 = -4ax$ where x_a is the first negative zero of the Tricomi confluent hypergeometric function*

$$x \mapsto U(x+1, 1+\frac{d}{2}, a\,r^2)\,,$$

which is not easy to calculate. Nevertheless the following figures 1 to 3 - done by simulations using Mathematica9 and the built-in functions FindRoot *and* HypergeometricU *for $d = 2$ and $r = 1$ - lead to conjecture that $a \mapsto x_a$ is bounded, and therefore the spectral gap of the process z is sublinear in a. In the figure 2 the blue curve, being the upper most one, corresponds to the function $a \mapsto x_a$. Scrolling the picture from up to down, one meets the curve corresponding to the second negative zero and so on.*

Remark 2.6. *It is well known that $\Phi(z) = |z|^2$ plays the role of a Lyapunov function for the Ornstein-Uhlenbeck process. But for the radial Ornstein-Uhlenbeck process $y(.)$ the situation is still better. Indeed its infinitesimal generator denoted by $\mathcal{L}$ is given, for $\rho > r/2$, by*

$$\mathcal{L}g(\rho) = \frac{1}{4}\,g''(\rho) - \left(2a\rho - \frac{d-1}{4\rho}\right)g'(\rho)\,,$$

so that, if $g(\rho)=\varphi(\rho)=\rho^2$, *it holds*

$$\mathcal{L}\varphi(\rho)=\frac{d}{2}-4a\rho^2 \qquad \text{provided } \rho>r/2\,,$$

so that $\mathcal{L}\varphi\le -4a\varepsilon\,\varphi$ *as soon as* $\frac{1}{1-\varepsilon}\le\frac{2ar^2}{d}$ *for some* $\varepsilon>0$.
It follows that the process $t\mapsto e^{4a\varepsilon t}y^2(t)$ *is a supermartingale up to the first time* τ_r *the process* y *hits the value* $r/2$. *This yields*

Proposition 2.7. *Assume that* $a>\frac{d}{2r^2}$ *and define* $\tau_r=\inf\{t; y(t)=|Y(t)|=r/2\}$ *the hitting time of a packing configuration. Then*

$$\mathsf{P}(\tau_r>t)\le\frac{4\mathsf{E}(|Y(0)|^2)}{r^2}\,e^{-4(a-\frac{d}{2r^2})t}\,.$$

This means that the system reaches a packing configuration $y=|Y|=r/2$ *before time* t *with a probability at least equal to* $1-\frac{4\mathsf{E}(|Y(0)|^2)}{r^2}\,e^{-4(a-\frac{d}{2r^2})t}$. *This statement should be particularly interesting to generalize to a higher number of balls.*

In the next section we shall look at the case of three hard balls, where real difficulties begin to occur.

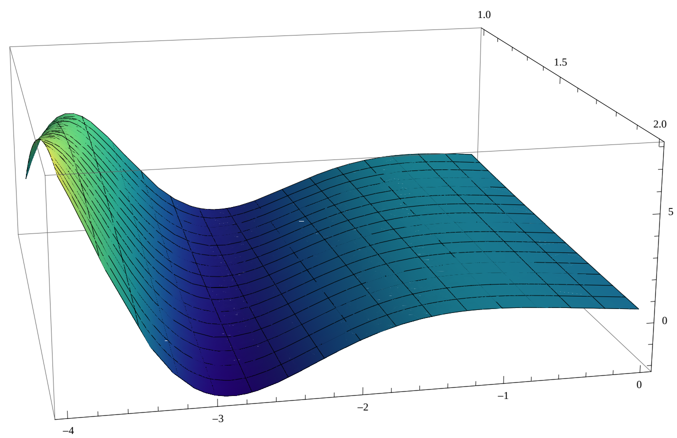

Figure 1. 3-dimensional plot of the function $(a,\mathrm{x})\mapsto U(\mathrm{x}+1,2,a)$.

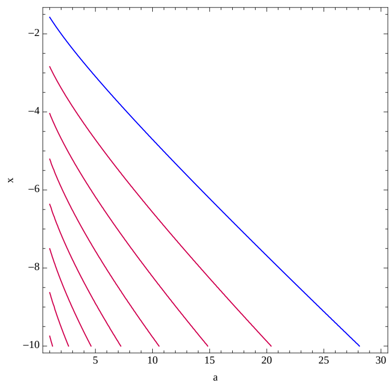

Figure 2. The zeros of the function $(a, \mathrm{x}) \mapsto U(\mathrm{x}+1, 2, a)$.

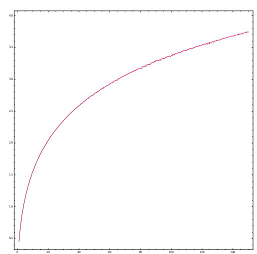

Figure 3. The logarithm of $-x_a$ as function of a.

3. The case of three balls.

We now address the case of three Brownian hard balls with attractive interaction. That is, we consider the dynamics (A) and (B) with $n = 3$ and $a > 0$. For simplicity, from now on we assume that $r = 1$. Our aim is to prove Theorem 1.9 for $n = 3$. Recall that the relative positions

$$Y_i = X_i - (X_1 + X_2 + X_3)/3$$

of the three hard balls follow the dynamics

$$\text{(B)} \begin{cases} \text{for } i \in \{1,2,3\}\,,\ t \in \mathbb{R}^+, \\ Y_i(t) = Y_i(0) + W_i(t) - \overline{W}(t) - a \displaystyle\sum_{j=1}^{3} \int_0^t (Y_i(s) - Y_j(s))ds + \sum_{j=1}^{3} \int_0^t (Y_i(s) - Y_j(s))dL_{ij}(s) \\ L_{ij}(0) = 0, \quad L_{ij} \equiv L_{ji} \quad \text{and} \quad L_{ij}(t) = \displaystyle\int_0^t \mathbb{1}_{|Y_i(s)-Y_j(s)|=1}\, dL_{ij}(s), \quad L_{ii} \equiv 0. \end{cases}$$

where W_1, W_2, W_3 are independent standard d-dimensional Brownian motions and $\overline{W} := (W_1+W_2+W_3)/3$. We noted in section 1 that the process Y is a $\overline{D'}$-valued Feller process satisfying the local Dobrushin condition, where

$$D' = \{\mathbf{y} \in (\mathbb{R}^d)^3\,;\ |y_1 - y_2| > 1\,, |y_2 - y_3| > 1\,, |y_3 - y_1| > 1 \ \text{and}\ y_1 + y_2 + y_3 = 0\}\,.$$

The invariant probability measure for the system (B) is given by

$$d\pi_a(\mathbf{y}) = Z_a^{-1}\, e^{-\frac{a}{2}V(\mathbf{y})}\, \mathbb{1}_{D'}(\mathbf{y})\, d\mathbf{y}$$

where Z_a is the normalization constant. The function $V(\mathbf{y}) = |y_1 - y_2|^2 + |y_2 - y_3|^2 + |y_3 - y_1|^2$ is, as before, the quadratic energy of the system. The configurations with minimal quadratic energy are triangular packings and build the set

$$\mathbf{E}_{min} = \{\mathbf{y} \in D' \,;\, V(\mathbf{y}) = 3\}\,.$$

We will prove in the sequel the following theorem.

Theorem 3.1. *Let Y satisfying* (B) *be the process of the relative positions of three hard balls and let $(P_t)_t$ be its transition distribution. There exists $\beta > 0$ such that*

$$\forall \mathbf{y} \in D' \quad \| P_t(\mathbf{y}, .) - \pi_a \|_{TV} \leq C(\mathbf{y})\, e^{-\beta t}\,.$$

In the large attraction regime, we obtain asymptotically in time the concentration of the system around packing triangular configurations in the sense that, for all $\mathbf{y} \in D'$,

$$\forall \varepsilon, \eta > 0, \exists a_0, t_0 \text{ s.t. } a > a_0 \text{ and } t > t_0 \Rightarrow \mathsf{P}(dist(Y(t), \mathbf{E}_{min}) \leq \eta \; |Y(0) = \mathbf{y}) \geq 1 - \varepsilon. \tag{3.2}$$

***Proof of* (3.2).** Let $\mathbf{E}^\eta_{min} = \{\mathbf{y} \in D' \,;\, V(\mathbf{y}) \leq 3 + \eta\}$ be the set of configurations with η-minimal energy. Clearly, $\pi_a(\mathbf{E}^\eta_{min})$ is large for a large enough:

$$\forall \varepsilon > 0 \quad \exists a_0 \text{ s.t. } \forall a > a_0 \quad \pi_a(\mathbf{E}^\eta_{min}) \geq 1 - \varepsilon\,.$$

because

$$\pi_a((\mathbf{E}^\eta_{min})^c) = \frac{\int_{D'} e^{-\frac{a}{2}V(\mathbf{y})}\, \mathbb{1}_{V(\mathbf{y})>3+\eta}\; d\mathbf{y}}{\int_{D'} e^{-\frac{a}{2}V(\mathbf{y})}\; d\mathbf{y}} \leq \frac{\int_{D'} e^{-\frac{a}{2}(V(\mathbf{y})-3-\eta)}\, \mathbb{1}_{V(\mathbf{y})>3+\eta}\; d\mathbf{y}}{\int_{D'} e^{-\frac{a}{2}(V(\mathbf{y})-3-\eta)}\, \mathbb{1}_{V(\mathbf{y})\leq 3+\eta}\; d\mathbf{y}}$$

hence

$$\pi_a((\mathbf{E}^\eta_{min})^c) \leq \frac{1}{\int_{D'} \mathbb{1}_{V(\mathbf{y})\leq 3+\eta}\; d\mathbf{y}} \int_{D'} e^{-\frac{a}{2}(V(\mathbf{y})-3-\eta)}\, \mathbb{1}_{V(\mathbf{y})>3+\eta}\; d\mathbf{y}$$

which vanishes for a tending to infinity. On the other side, the convergence in total variation implies

$$\lim_{t\to+\infty} \mathsf{P}(\; Y(t) \in \mathbf{E}^\eta_{min} \; |Y(0) = \mathbf{y}) = \pi_a(\mathbf{E}^\eta_{min})\,.$$

Using the continuity of V, this yields (3.2).
The same technics obviously works for any number n of balls. □

The technics we will use in order to prove the main part of the above Theorem, i.e. the exponential ergodicity, is very intricate. It relies on hitting time estimates and is the subject of the rest of the paper.

3.1. Quadratic energy and hitting time of clusters.

In this Section 3.1 we present the energy as a Lyapounov function, define the compact set of cluster patterns and state that Y satisfies the assumptions of Theorem 1.10.
For a configuration $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^{3d}$, or equivalently for a pattern $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^{3d}$ of relative positions (with $y_i = x_i - (x_1 + x_2 + x_3)/3$), recall that the quadratic (total) energy satisfies

$$V(\mathbf{y}) = 3(|y_1|^2 + |y_2|^2 + |y_3|^2) = |x_1 - x_2|^2 + |x_2 - x_3|^2 + |x_3 - x_1|^2.$$

Definition 3.3. *Fix $R > 0$. We say that a relative position $\mathbf{y} = (y_1, y_2, y_3) \in \mathbb{R}^{3d}$ forms an R-*cluster*, if there exists a permutation σ on $\{1, 2, 3\}$ such that*

$$|y_{\sigma(1)} - y_{\sigma(2)}|^2 \le 1 + R \quad \text{and} \quad |y_{\sigma(2)} - y_{\sigma(3)}|^2 \le 1 + R,$$

or equivalently,

$$\forall i \in \{1, 2, 3\}, \quad \exists j \neq i, \quad |y_i - y_j|^2 \le 1 + R.$$

Note that

$$K_R := \{\mathbf{y} \in (\mathbb{R}^d)^3\, ; \, \mathbf{y} \text{ forms an } R\text{-cluster}\}$$

is a compact set of $\mathbb{R}^{3d}$.
In order to check the assumptions of Theorem 1.10, we would like to control the time needed by the process Y in such a way that its value forms an R-cluster.

Proposition 3.4. *The relative positions process Y has the following properties :*

1. *Its Lyapounov quadratic energy V fulfills*

$$\forall \mathbf{y} \in D' \quad V(\mathbf{y}) \ge 3 \quad \text{and} \quad \lim_{|\mathbf{y}| \to +\infty} V(\mathbf{y}) = +\infty. \tag{3.5}$$

2. *The K_R-hitting time*

$$\tau := \inf\{t \ge 0, Y(t) = (Y_1(t), Y_2(t), Y_3(t)) \text{ forms an } R\text{-cluster }\}$$

satisfies inequalities

$$\forall \mathbf{y} \in D' \quad \mathsf{E}_{\mathbf{y}} \left(e^{\lambda\tau} V(Y(\tau))\right) \le V(\mathbf{y}) \tag{3.6}$$

and

$$\forall \mathbf{y} \in D' \quad \mathsf{E}_{\mathbf{y}} \left(V(Y(t)) 1\!\!1_{\tau > t}\right) \le 2e^{-\lambda t} V(\mathbf{y}) \tag{3.7}$$

for

$$\lambda = \min(a, a^2) \quad \text{and any} \quad R \ge \frac{48a + 16d + 60}{a} e^{10505/a}. \tag{3.8}$$

3. *The quadratic energy of the system is uniformly bounded in time for any initial R-cluster position, i.e. for R as above*

$$\sup_{\mathbf{y} \in K_R} \sup_{t > 0} \mathsf{E}_{\mathbf{y}} \left(V(Y(t))\right) < +\infty \tag{3.9}$$

Proof of Proposition 3.4.
The properties (3.5) are an obvious consequence of the definition of V.
The proofs of (3.6) and (3.7) rely on a study of the length of the largest median in the triangle of particles, as a funtion of the time. These proofs are postponed to section 3.2.
In order to prove (3.9) we first establish the following consequence of (3.7)

$$\exists T > 0 \quad \exists D_1 > 0 \quad \text{s.t.} \quad \sup_{\mathbf{y} \in K_R} \sup_{t \in [0;T]} \mathsf{E}_{\mathbf{y}} \left(V(Y(t))\right) \le D_1. \tag{3.10}$$

Take R as in Theorem 3.4 part (2) and take some larger $\bar{R} > R$. Construct a sequence of stopping times $\xi_k, k \ge 0$ in the following way. Assume that $Y(0) = \mathbf{y} \in K_R$ and define $\xi_0 = 0$,

$$\xi_{2j-1} = \inf\{t > \xi_{2j-2} : Y(t) \notin K_{\bar{R}}\}, \quad \xi_{2j} = \inf\{t > \xi_{2j-1} : Y(t) \in K_R\}, \quad j \ge 1.$$

Then

$$\mathsf{E}_{\mathbf{y}}\left(V(Y(t))\right) = \sum_{k=1}^{\infty} \mathsf{E}_{\mathbf{y}}\left(V(Y(t))\mathbb{1}_{t\in[\xi_{k-1},\xi_k)}\right) =: \sum_{k \text{ is even}} + \sum_{k \text{ is odd}} .$$

Let

$$\|V\|_{K_{\bar{R}}} := \max_{\mathbf{y}\in K_{\bar{R}}} V(\mathbf{y}) = 6(1+\bar{R}).$$

When k is odd and $t \in [\xi_{k-1}, \xi_k)$, we have $Y(t) \in K_{\bar{R}}$, which means that

$$\sum_{k \text{ is odd}} \le \|V\|_{K_{\bar{R}}}.$$

When k is even, we have

$$\mathsf{E}_{\mathbf{y}}\left(V(Y(t))\mathbb{1}_{t\in[\xi_{k-1},\xi_k)}\right) \le \mathsf{E}_{\mathbf{y}}\left(\mathbb{1}_{t\ge\xi_{k-1}}\left(\mathsf{E}[V(Y(t))\mathbb{1}_{\xi_k>t}|\mathcal{F}_{\xi_{k-1}}]\right)\right).$$

Note that, by the continuity of trajectories, $Y(\xi_{k-1}) \in K_{\bar{R}}$. Then, applying the strong Markov property at the time moment ξ_{k-1} and (3.7), we get

$$\mathsf{E}_{\mathbf{y}}\left(V(Y(t))\mathbb{1}_{t\in[\xi_{k-1},\xi_k)}\right) \le 2\|V\|_{K_{\bar{R}}}\mathsf{P}_{\mathbf{y}}(t \ge \xi_{k-1}).$$

Hence

$$\sum_{k \text{ is even}} \le 2\|V\|_{K_{\bar{R}}} \sum_{k \text{ is even}} \mathsf{P}_{\mathbf{y}}(t \ge \xi_{k-1}),$$

and to prove (3.10) it is enough to prove that for some T

$$\sup_{\mathbf{y}\in K_R}\sup_{t\le T}\sum_{k \text{ is even}} \mathsf{P}_{\mathbf{y}}(t \ge \xi_{k-1}) < \infty.$$

By the Chebyshev-Markov inequality, for any fixed $c > 0$ and $T > 0$

$$\mathsf{P}_{\mathbf{y}}(t \ge \xi_{k-1}) \le e^{ct}\mathsf{E}_{\mathbf{y}}(e^{-c\xi_{k-1}}) \le e^{cT}\mathsf{E}_{\mathbf{y}}(e^{-c\xi_{k-1}}), \quad \forall t \le T.$$

Clearly, the exponential moment $\mathsf{E}_{\mathbf{y}}(e^{-c\xi_{k-1}})$ can be expressed iteratively via the conditional exponential moments of the differences $\xi_j - \xi_{j-1}$ w.r.t. $\mathcal{F}_{\xi_{j-1}}$, $j = 1, \dots, k-1$. When j is odd, this conditional exponential moment can be estimated as follows:

$$\mathsf{E}_{\mathbf{y}}\left[e^{-c(\xi_j-\xi_{j-1})}\Big|\mathcal{F}_{\xi_{j-1}}\right] \le \sup_{\mathbf{y}\in K_R}\mathsf{E}_{\mathbf{y}}(e^{-c\varsigma}), \quad \varsigma = \inf\{t : X(t) \notin K_{\bar{R}}\}.$$

Note that

$$q := \sup_{\mathbf{y}\in K_R}\mathsf{E}_{\mathbf{y}}(e^{-c\varsigma}) < 1$$

because otherwise, by the Feller property of the process Y, there would exist $\mathbf{y} \in K_R$ such that $\varsigma = 0$ $\mathsf{P}_{\mathbf{y}}$-a.s, which would contradict the continuity of the trajectories of Y. Then

$$\mathsf{E}_{\mathbf{y}}(e^{-c\xi_{k-1}}) \le q^{k/2}$$

$$\sup_{\mathbf{y}\in K_R}\sup_{t\le T}\sum_{k \text{ is even}} \mathsf{P}_{\mathbf{y}}(t \ge \xi_{k-1}) \le e^{cT}\sum_{k \text{ is even}} q^{k/2} < \infty,$$

which completes the proof of (3.10).

Let us now deduce the uniform bound (3.9) from the finite time bound (3.10). This proof is simple and similar to that of Lemma A.4 in A. M. Kulik (2011). Indeed, let τ' be the first time moment for $X(t)$ to form an R-cluster after T

$$\tau' := \inf \{ \ t \geq T \text{ s.t. } Y(t) \text{ forms an } R\text{-cluster } \}$$

Then for $t > T$

$$\mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \right) = \mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \mathbb{1}_{\tau' > t} \right) + \mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \mathbb{1}_{\tau' \leq t} \right).$$

We have by the Markov property of Y, for λ small enough and R large enough for (3.7) to hold

$$\mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \mathbb{1}_{\tau' > t} \right) = \int_{\mathbb{R}^{3d}} \Big(\mathsf{E}_{\mathbf{x}} \left(V(Y(t-T)) \mathbb{1}_{\tau > t-T} \right) \Big) \mathsf{P}_T(\mathbf{y}, d\mathbf{x}) \leq 2e^{-\lambda(t-T)} \mathsf{E}_{\mathbf{y}} \left(V(Y(T)) \right).$$

On the other hand, by the strong Markov property of Y, we have

$$\mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \mathbb{1}_{\tau' \leq t} \right) = \mathsf{E}_{\mathbf{y}} \left(\left(\mathsf{E}_{Y(\tau')} V(Y(t - \tau')) \right) \mathbb{1}_{\tau' \leq t} \right) \leq \sup_{\mathbf{y} \in K_R} \sup_{s \leq t-T} \mathsf{E}_{\mathbf{y}} \left(V(Y(s)) \right).$$

Let $D_k = \sup_{\mathbf{y} \in K_R} \sup_{t \leq kT} \mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \right)$. Then the above estimates and (3.10) yield for every $\mathbf{y} \in K_R$ and $(k-1)T \leq t \leq kT$

$$\mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \right) \leq 2e^{-\lambda(k-2)T} \sup_{\mathbf{y} \in K_R} \mathsf{E}_{\mathbf{y}}(V(X(T))) + D_{k-1} \leq 2e^{-\lambda(k-2)T} D_1 + D_{k-1}.$$

Then

$$D_k = \max \left(D_{k-1}, \sup_{\mathbf{y} \in K_R} \sup_{(k-1)T \leq t \leq kT} \mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \right) \right) \leq 2e^{-\lambda(k-2)T} D_1 + D_{k-1},$$

and consequently

$$\sup_{\mathbf{y} \in K_R} \sup_{t \geq 0} \mathsf{E}_{\mathbf{y}} \left(V(Y(t)) \right) \leq D_1 + D_1 \sum_{k=2} 2e^{-\lambda(k-2)T} = D_1 + 2D_1[1 - e^{-\lambda T}]^{-1} < \infty.$$

□

3.2. Cluster hitting time estimates.

This section is devoted to the proof of assertion (2) of Proposition 3.4, that is (3.6) and (3.7). It will complete the proof of Theorem 3.4. From now on, R and λ are fixed parameters. If the starting configuration $Y(0) = \mathbf{y} \in \mathbb{R}^{3d}$ already forms an R-cluster, $\tau = 0$ and (3.6), (3.7) are trivial. In the sequel, we assume that $\mathbf{y} \notin K_R$.

Let us reduce the problem to the study of the time the farthest ball need to come closer to the others. Since $Y(0) = \mathbf{y} \notin K_R$, some ball $\mathbf{i}$ has a center y_i which is farther than $\sqrt{1+R}$ from the other two centers. Suppose, for instance, $\mathbf{i} = \mathbf{1}$. We construct a sequence of stopping times corresponding to hitting times of levels for the distance between the balls $\mathbf{2}$ and $\mathbf{3}$. These levels depend on two parameters $\delta > \delta' > 0$ which will be chosen later. Put $\sigma_0 = 0$, and for $k \geq 1$,

$$\sigma_{2k-1} := \inf \left\{ t > \sigma_{2k-2} : |Y_2 - Y_3|^2 \leq 1 + 2\delta' \right\},$$
$$\sigma_{2k} := \inf \left\{ t > \sigma_{2k-1} : |Y_2 - Y_3|^2 \geq 1 + 2\delta \right\},$$

and define a *time-depending border level* as

$$R(t) = \sum_{k=1}^{\infty} \Big[R\mathbb{1}_{t\in[\sigma_{2k-2},\sigma_{2k-1})} + R'\mathbb{1}_{t\in[\sigma_{2k-1},\sigma_{2k})} \Big], \quad t \in [0,\infty).$$

for some $0 < R' < R$. Let us now define the first time the ball number **1** comes closer than $R(.)$ to one of the others

$$\tau_1 := \inf\{t \geq 0 \; ; \; \min(|Y_1(t) - Y_2(t)|^2, |Y_1(t) - Y_3(t)|^2) \leq 1 + R(t)\}.$$

Consider the configuration $Y(\tau_1)$. If it forms an R-cluster, then put $\tau_2 = \tau_1$. Otherwise, there exists some ball whose center is farther than $\sqrt{1+R}$ from the other two centers. Since $R(t) \leq R$, this ball should be either **2** or **3**. Define then the new sequence of stopping times corresponding to level hitting times of the distance between the other two balls (with the same values δ, δ') and the corresponding time-depending border level and respective τ_2, and so on. By monotonicity, there exists an a.s. limit, $\tau_\infty = \lim_n \tau_n$.
To obtain the desired estimates on the K_R-hitting time for Y, we only have to prove the following proposition.

Proposition 3.11. *If R and λ are as in (3.8), for any starting configuration $\mathbf{y} \in D'$*

$$\mathsf{E}_{\mathbf{y}} \left(e^{\lambda\tau_1} V(Y(\tau_1)) \right) \leq V(\mathbf{y})$$

and for any $\mathbf{y} \in D'$ and any finite time horizon $T \in \mathbb{R}^+$

$$\mathsf{E}_{\mathbf{y}} \left(e^{\lambda\tau_1} V(Y(\tau_1)) + e^{\lambda(\tau_1\wedge T)} V(Y(\tau_1 \wedge T)) \right) \leq 2V(\mathbf{y})$$

Remark that Proposition 3.11 implies Proposition 3.4 part (2)
Indeed, by construction and by the strong Markov property of $Y(t), t \geq 0$, it follows from Proposition 3.11 and its analogous for $\tau_2 - \tau_1, \tau_3 - \tau_2, \dots,$ that

$$\mathsf{E}_{\mathbf{y}} \left(e^{\lambda\tau_n} V(Y(\tau_n)) \right) \leq V(\mathbf{y}), \quad n \geq 1. \tag{3.12}$$

Because V is bounded from below, this implies that $\tau_\infty < \infty$ a.s. On the other hand, by the construction and by the continuity of the trajectories of $Y(t), t \geq 0$ it is easy to see that $Y(\tau_\infty)$ forms an R-cluster as soon as $\tau_\infty < \infty$, and consequently $\tau \leq \tau_\infty$ a.s. Hence, (3.6) follows from (3.12) by the Fatou lemma.
By Fatou lemma again , the second inequality in Proposition 3.11 implies

$$\mathsf{E}_{\mathbf{y}} \left(e^{\lambda(\tau_\infty\wedge T)} V(Y(\tau_\infty \wedge T)) \right) \leq 2V(\mathbf{y})$$

Since $\tau_\infty \wedge T \geq \tau \wedge T$ this gives

$$\mathsf{E}_{\mathbf{y}} \left(e^{\lambda(\tau\wedge T)} V(Y(\tau_\infty \wedge T)) \right) \leq 2V(\mathbf{y})$$

which implies (3.7) because $e^{\lambda T}\mathbb{1}_{\tau>T} \leq e^{\lambda(\tau\wedge T)}$ and $\mathbb{1}_{\tau>T} V(Y(\tau_\infty \wedge T)) = \mathbb{1}_{\tau>T} V(Y(T))$.

Our aim now is to prove Proposition 3.11.

3.3. Dynamics of the medians of the triangle.

Let us introduce the following vectors describing the triangle Y_1, Y_2, Y_3:

$$U_1 := \sqrt{\frac{2}{3}}\left(\frac{Y_2+Y_3}{2} - Y_1\right), \quad U_{23} := \frac{1}{\sqrt{2}}(Y_2 - Y_3).$$

U_1 is the (scaled) median starting from Y_1 and U_{23} is its (scaled) opposite side.
In order to prove Proposition 3.11, we only have to consider the behaviour of Y up to time τ_1. But, before the time moment τ_1, the ball **1** does not hit any other ball. Therefore, on the (random) time interval $[0, \tau_1]$ processes U_1, U_{23} satisfy the following simple SDE's:

$$\begin{cases} dU_1(t) &= dB_1(t) - 3aU_1(t)\,dt, \\ dU_{23}(t) &= dB_{23}(t) - 3aU_{23}(t)\,dt + 2U_{23}(t)\,dL_{23}(t). \end{cases} \tag{3.13}$$

Note that the martingale terms B_1, B_{23} are independent $\mathbb{R}^d$-valued Brownian motions and that the dynamics of the median U_1 does not include a local time term up to time τ_1.
Also note that the quadratic energy has a simple expression as a function of the median and its opposite side, and that these two lengths control the size of the triangle Y.

Lemma 3.14.

$$V(Y) = 3(|U_1|^2 + |U_{23}|^2)$$

and for $j = 2$ or $j = 3$

$$\frac{1}{3}|Y_2 - Y_1|^2 + \frac{1}{3}|Y_3 - Y_1|^2 - \frac{1}{3}|U_{23}|^2 = |U_1|^2 \le \frac{4}{3}|Y_j - Y_1|^2 + \frac{2}{3}|U_{23}|^2 \tag{3.15}$$

Proof. The equalities are simple norm computations and $U_1 = \frac{1}{\sqrt{6}}((Y_3 - Y_2) - 2(Y_1 - Y_2))$ i.e. $|U_1|^2 = \frac{1}{6}|2(Y_2 - Y_1) - \sqrt{2}U_{23}|^2$ gives the upper bound for $j = 2$. □

3.4. The time weighted energy decreases.

Define the time weighted energy of the system by

$$H(t) := e^{\lambda t} V(Y(t)) = 3e^{\lambda t}|U_1(t)|^2 + 3e^{\lambda t}|U_{23}(t)|^2$$

and compute its mean value $\mathsf{E}_{\mathbf{y}}\left(H(\zeta_k)\right)$ at the random time $\zeta_k := T \wedge \tau_1 \wedge \sigma_k$ for a fixed time horizon T which may be finite ($T \in \mathbb{R}^+$ and $\zeta_k := T \wedge \tau_1 \wedge \sigma_k$) or infinite ($T = +\infty$ and $\zeta_k := \tau_1 \wedge \sigma_k$).

$$\begin{aligned} &\mathsf{E}_{\mathbf{y}}\left(H(T\wedge\tau_1\wedge\sigma_k) + H(\tau_1\wedge\sigma_k)\right) \\ &= \mathsf{E}_{\mathbf{y}}\left((H(T\wedge\tau_1\wedge\sigma_k) + H(\tau_1\wedge\sigma_k))\mathbb{1}_{T\wedge\tau_1>\sigma_{k-1}}\right) \\ &\quad + \mathsf{E}_{\mathbf{y}}\left((H(T\wedge\tau_1\wedge\sigma_{k-1}) + H(\tau_1\wedge\sigma_k))\mathbb{1}_{\tau_1>\sigma_{k-1}\ge T}\right) \\ &\quad + \mathsf{E}_{\mathbf{y}}\left((H(T\wedge\tau_1\wedge\sigma_{k-1}) + H(\tau_1\wedge\sigma_{k-1}))\mathbb{1}_{\sigma_{k-1}\ge\tau_1}\right) \end{aligned} \tag{3.16}$$

But

$$\mathsf{E}_{\mathbf{y}}\left(H(T\wedge\tau_1\wedge\sigma_k)\mathbb{1}_{T\wedge\tau_1>\sigma_{k-1}}\right)=\mathsf{E}_{\mathbf{y}}\left(\mathbb{1}_{T\wedge\tau_1>\sigma_{k-1}}e^{\lambda\sigma_{k-1}}\mathsf{E}_{\mathbf{y}}\left[H^{\sigma_{k-1}}(T\wedge\tau_1\wedge\sigma_k)\Big|\mathcal{F}_{\sigma_{k-1}}\right]\right)$$

where

$$H^\sigma(t):=e^{\lambda(t-\sigma\wedge t)}V(Y(t)).$$

Proposition 3.17. *Under a proper choice of λ, R' and R, for every $k\geq 1$*

$$\mathsf{E}_{\mathbf{y}}\left[H^{\sigma_{k-1}}(\tau_1\wedge\sigma_k)\Big|\mathcal{F}_{\sigma_{k-1}}\right]\leq H^{\sigma_{k-1}}(\sigma_{k-1})\qquad\text{on the set }\{\tau_1>\sigma_{k-1}\}.\tag{3.18}$$

and for each finite time horizon $T\in\mathbb{R}^+$

$$\mathsf{E}_{\mathbf{y}}\left[H^{\sigma_{k-1}}(T\wedge\tau_1\wedge\sigma_k)+H^{\sigma_{k-1}}(\tau_1\wedge\sigma_k)\Big|\mathcal{F}_{\sigma_{k-1}}\right]\leq 2H^{\sigma_{k-1}}(\sigma_{k-1})\qquad\text{on the set }\{T\wedge\tau_1>\sigma_{k-1}\}.\tag{3.19}$$

Once this lemma is proven, by (3.16) we will have

$$\mathsf{E}_{\mathbf{y}}\left(H(\tau_1\wedge\sigma_k)\right)\leq\mathsf{E}_{\mathbf{y}}\left(H(\tau_1\wedge\sigma_{k-1})\right))$$

$$\mathsf{E}_{\mathbf{y}}\left(H(T\wedge\tau_1\wedge\sigma_k)+H(\tau_1\wedge\sigma_k)\right)\leq\mathsf{E}_{\mathbf{y}}\left((H(T\wedge\tau_1\wedge\sigma_{k-1})+H(\tau_1\wedge\sigma_{k-1})\right))$$

and iterating these inequalities we obtain Proposition 3.11 because $H(0)=V(\mathbf{y})$. In order to prove Proposition 3.17 we have to consider two cases.

Proof of Proposition 3.17 when k is odd (i.e. $|U_{23}|^2$ goes downhill).
Suppose $\tau_1>\sigma_{k-1}$ and look at the dynamics during the interval $[\sigma_{k-1},\sigma_k\wedge\tau_1)$. This case is simple because no balls can collide, hence the local time term L_{23} in (3.13) vanishes. Therefore by (3.13) we have, on this time interval,

$$\begin{aligned}dH^{\sigma_{k-1}}(t)&=6e^{\lambda(t-\sigma_{k-1})}(U_1(t),dW_1(t))+6e^{\lambda(t-\sigma_{k-1})}(U_{23}(t),dW_{23}(t))\quad\text{(martingale part)}\\&+\ H^{\sigma_{k-1}}(t)\big(\frac{6d}{V(Y(t-\sigma_{k-1}))}+\lambda-6a\big)\,dt;\end{aligned}$$

On $[\sigma_{k-1},\sigma_k\wedge\tau_1)$ the border level $R(t)$ equals R, so we have $V(Y(t))>2(R+1)+2(\frac{1}{2}+\delta')=2R+3+2\delta'$. Therefore

$$\begin{aligned}\mathsf{E}_{\mathbf{y}}\left[H^{\sigma_{k-1}}(\tau_1\wedge\sigma_k)\Big|\mathcal{F}_{\sigma_{k-1}}\right]&\leq H^{\sigma_{k-1}}(\sigma_{k-1})\\&+\left(\frac{6d}{2R+3+2\delta'}+\lambda-6a\right)\mathsf{E}_{\mathbf{y}}\Big[\int_{\sigma_{k-1}}^{\tau_1\wedge\sigma_k}H^{\sigma_{k-1}}(t)\,dt\Big|\mathcal{F}_{\sigma_{k-1}}\Big].\end{aligned}$$

Since $H^{\sigma_{k-1}}(\sigma_{k-1})=V(Y(\sigma_{k-1}))$, this yields (3.18) provided that

$$\lambda\leq 6a-\frac{6d}{2R+3+2\delta'}.\tag{3.20}$$

Note that for any fixed time horizon $T\in\mathbb{R}^+$, the above calculation also holds with τ_1 replaced by $\tau_1\wedge T$. $\square$

Proof of Proposition 3.17 when k is even (i.e. $|U_{23}|^2$ goes uphill).
We look at the dynamics during the interval $[\sigma_{k-1}, \sigma_k \wedge \tau_1)$ again.
Up to a martingale term, $H^{\sigma_{k-1}}(\tau_1 \wedge \sigma_k) - H^{\sigma_{k-1}}(\sigma_{k-1})$ is equal to

$$(\lambda - 6a)\int_{\sigma_{k-1}}^{\tau_1\wedge\sigma_k} 3e^{\lambda(s-\sigma_{k-1})}|U_1(s)|^2\,ds + \frac{3d}{\lambda}\left(e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1\right) + 3e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}|U_{23}(\tau_1\wedge\sigma_k)|^2 - 3|U_{23}(\sigma_{k-1})|^2.$$

In this case, on $[\sigma_{k-1}, \sigma_k \wedge \tau_1)$, thanks to (3.15), $|U_1(s)|^2 \geq \frac{1}{3}(1+R') - \frac{1}{3}(\frac{1}{2}+\delta)$.
Moreover $|U_{23}^{\sigma_{k-1}}(\sigma_{k-1})|^2 = \frac{1}{2}+\delta'$ and $|U_{23}^{\sigma_{k-1}}(\tau_1\wedge\sigma_k)|^2 \leq \frac{1}{2}+\delta$.
Thus for any $\lambda < 6a$

$$\begin{aligned}
& \mathsf{E}_{\mathbf{y}}\left(H^{\sigma_{k-1}}(\tau_1\wedge\sigma_k) - H^{\sigma_{k-1}}(\sigma_{k-1})\Big|\mathcal{F}_{\sigma_{k-1}}\right)\\
\leq\ & \left((\lambda-6a)\left(\frac{1}{2}+R'-\delta\right)+3d\right)\mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1}{\lambda}\Big|\mathcal{F}_{\sigma_{k-1}}\right)\\
+\ & 3(\frac{1}{2}+\delta)\mathsf{E}_{\mathbf{y}}\left(e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}\Big|\mathcal{F}_{\sigma_{k-1}}\right) - 3(\frac{1}{2}+\delta')\\
=\ & (R'(\lambda-6a)+2\lambda(1+\delta)-3a(1-2\delta)+3d)\,\mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1}{\lambda}\Big|\mathcal{F}_{\sigma_{k-1}}\right)+3(\delta-\delta')
\end{aligned} \tag{3.21}$$

The key point in the whole proof is the fact that, under an appropriate choice of the parameters, this last expectation is finite and admits a uniform lower bound:

Lemma 3.22. *There exists $\underline{C}$ depending only on δ, δ' such that for each even k and for λ small enough*

$$0 < \underline{C} \leq \mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1}{\lambda}\Big|\mathcal{F}_{\sigma_{k-1}}\right) < +\infty$$

Once this lemma is proved, there will be an R' large enough (hence an R large enough) for

$$(R'(\lambda-6a)+2\lambda(1+\delta)-3a(1-2\delta)+3d)\underline{C}+3(\delta-\delta') \leq 0 \tag{3.23}$$

to hold and this will imply for $\tau_1 > \sigma_{k-1}$

$$\mathsf{E}_{\mathbf{y}}\left(H^{\sigma_{k-1}}(\tau_1\wedge\sigma_k) - H^{\sigma_{k-1}}(\sigma_{k-1})\Big|\mathcal{F}_{\sigma_{k-1}}\right) \leq 0 \tag{3.24}$$

which rewrites into 3.18.
Note that, as in the previous case, calculation (3.21) also holds with τ_1 replaced by $\tau_1 \wedge T$ for any fixed time horizon $T \in \mathbb{R}^+$. Summing the expressions with and without finite time horizon, and using the lower bound 0 for $\mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k\wedge T-\sigma_{k-1})}-1}{\lambda}\Big|\mathcal{F}_{\sigma_{k-1}}\right)$, we obtain that on $T\wedge\tau_1 > \sigma_{k-1}$

$$\mathsf{E}_{\mathbf{y}}\left(H^{\sigma_{k-1}}(\tau_1\wedge\sigma_k) + H^{\sigma_{k-1}}(T\wedge\tau_1\wedge\sigma_k) - 2H^{\sigma_{k-1}}(\sigma_{k-1})\Big|\mathcal{F}_{\sigma_{k-1}}\right) \leq 0$$

as soon as

$$(R'(\lambda-6a)+2\lambda(1+\delta)-3a(1-2\delta)+3d)\underline{C}+6(\delta-\delta') \leq 0 \tag{3.25}$$

□

Clearly, we can forget about condition (3.23) as any set of parameter satisfying condition (3.25) will satisfy (3.23) too. From now on, our aim is to prove Lemma 3.22. The finiteness of $\mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1}{\lambda}\Big|\mathcal{F}_{\sigma_{k-1}}\right)$ is obtained in section 3.5 and the uniform lower bound is constructed in section 3.6.

3.5. Existence of exponential moments of $\tau_1 \wedge \sigma_k - \sigma_{k-1}$ for even k.

We need a proof that for small enough λ's the exponential moment $\mathsf{E}_{\mathbf{y}}\left(e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}\middle|\mathcal{F}_{\sigma_{k-1}}\right)$ is finite. We use a comparison argument. Since a similar comparison argument will be needed to obtain a lower bound on the exponential moment, we directly construct a double inequality, though an upper bound is sufficient for our purpose in this section.

3.5.1. *Comparison with the level hitting time of a simple reflected SDE*

Consider a Wiener process B^U, independent on W_1, such that

$$B^U(t) = \int_0^t \frac{1}{|U_{23}(s)|}(U_{23}(s), dB_{23}(s)), \quad t < \tau_1,$$

and the process U solution to the following one-dimensional SDE with reflection at the point $\frac{1}{2}$:

$$U(t) = |U_{23}(0)|^2 + \int_0^t (d - 6aU(s))\, ds + 2\int_0^t U^{1/2}(s)\, dB^U(s) + L^U(t).$$

Then the processes $|U_{23}(t)|^2$ and $U(t)$ coincide up to the time τ_1, and $L^U(t) = 2L_{23}(t)$ for $t < \tau_1$. It is sufficient to prove the finiteness of $\mathsf{E}_{\frac{1}{2}+\delta'}(e^{\lambda\sigma})$ where $\sigma = \inf\{t : U(t) = \frac{1}{2} + \delta\}$. We make a time change, i.e we put

$$\zeta_t = \int_0^t 4U(s)ds, \quad \chi_t = \inf\{r : \zeta_r \geq t\}, \quad \tilde{U}(t) = U(\chi_t).$$

Then $\tilde{U}$ satisfies the one-dimensional SDE with reflection at the point $\frac{1}{2}$

$$d\tilde{U}(t) = \frac{d - 6a\tilde{U}(t)}{4\tilde{U}(t)}\, dt + d\tilde{B}(t) + d\tilde{L}(t),$$

where $\tilde{B}$ is a Wiener process. Since $\frac{1}{2} \leq \tilde{U}(t) \leq \frac{1}{2} + \delta$ up to time σ, then

$$(2 + 4\delta)\sigma \geq \tilde{\sigma} := \zeta_\sigma = \inf\{t : \tilde{U}(t) = \frac{1}{2} + \delta\} \geq \sigma$$

Since the drift of $\tilde{U}(t)$ is bounded from above and from below by some constants

$$C_1 := -\frac{3}{2}a \leq \frac{d - 6a\tilde{U}(t)}{4\tilde{U}(t)} \leq \frac{d}{2} - \frac{3}{2}a < \frac{d}{2} =: C_2,$$

we can compare $\tilde{U}(t)$ with reflected Brownian motions with constant drifts C_1 and C_2, as in A. R. Ward and P.W. Glynn (2003) Proposition 2. We then obtain $\hat{U}_1 \leq \tilde{U} \leq \hat{U}_2$ where $\hat{U}_i, i = 1, 2$ satisfy the one-dimensional SDE's with reflection at the point $\frac{1}{2}$

$$d\hat{U}_i(t) = C_i dt + d\hat{B}(t) + d\hat{L}_i(t), \quad \hat{U}_i(0) = \frac{1}{2} + \delta'.$$

where $\hat{B}$ is an $\mathbb{R}$-valued brownian motion and $\hat{L}_i(t) = \int_0^t \mathbb{I}_{\hat{U}_i(s)=\frac{1}{2}} d\hat{L}_i st)$. This allows us to compare the δ-level hitting times of the three processes:

$$\hat{\sigma}_1 := \inf\{t : \hat{U}_1(t) \geq \frac{1}{2} + \delta\} \geq \tilde{\sigma} \geq \hat{\sigma}_2 := \inf\{t : \hat{U}_2(t) \geq \frac{1}{2} + \delta\}.$$

and we obtain

$$\frac{\hat{\sigma}_2}{2+4\delta} \leq \sigma \leq \hat{\sigma}_1. \tag{3.26}$$

In the sequel, we compute the exponential moments of hitting times $\hat{\sigma}_i, i \in \{1,2\}$. For the time being, we drop the indices on $\hat{\sigma}_i$, C_i and $\hat{L}_i$.

3.5.2. Exponential moments of level hitting times

It is equivalent to consider the hitting time of $\frac{1}{2} + \delta$ for a brownian motion starting from $\frac{1}{2} + \delta'$ with constant drift C and reflection at $1/2$ or to consider the hitting time of δ for a brownian motion starting from δ' with constant drift C and reflection at 0. Girsanov theorem for processes with reflection G.N. Kinkladze (2011) and Doob's optional sampling theorem implies that for all negative λ

$$\mathsf{E}_{\frac{1}{2}+\delta'}\left(e^{\lambda\hat{\sigma}}\right) = e^{C(\delta-\delta')}\mathsf{E}\left(e^{(\lambda-\frac{C^2}{2})\inf\{t;|\delta'+\hat{B}(t)|=\delta\}}e^{-\frac{C}{2}\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_0^t \mathbb{I}_{[0;\varepsilon[}(|\delta'+\hat{B}(t)|)}\right).$$

Suppose $C \neq 0$. Then using Formula 2.3.3 in A. Borodin and P. Salminen (2002) for $r = 0, x = \delta', z = \delta, \alpha = C^2/2 - \lambda, \gamma = C/2$, which holds for any $\lambda < C^2/2$,

$$\mathsf{E}_{\frac{1}{2}+\delta'}\left(e^{\lambda\hat{\sigma}}\right) = e^{C(\delta-\delta')}\frac{v\cosh(C\delta' v) + \sinh(C\delta' v)}{v\cosh(C\delta v) + \sinh(C\delta v)} =: \Psi(v) \tag{3.27}$$

where $v(\lambda) := \sqrt{1 - 2\frac{\lambda}{C^2}}$. This is an analytical function of λ thus the same formula holds for positive λ's, as long as $v(\lambda)$ is well defined and the denominator doesn't vanish. But any positive v such that the denominator vanishes satisfies $\frac{x\cosh(x)}{\sinh(x)} = -C\delta$ for $x = C\delta v$. Since the function $x \longmapsto \frac{x\cosh(x)}{\sinh(x)}$ is larger than 1 on the whole $\mathbb{R}$, the condition $-C\delta < 1$ ensures that the λ-exponential moment exists for positive λ's satisfying $\lambda < C^2/2$.
$\sigma \leq \tilde{\sigma} \leq \hat{\sigma}_1$ thus $\mathsf{E}_{\frac{1}{2}+\delta'}\left(e^{\lambda\sigma}\right)$ is finite as soon as $\lambda < C_1^2/2$ and $-C_1\delta < 1$, hence

$$\lambda < \frac{9}{8}a^2 \quad \text{and} \quad \delta < \frac{2}{3a} \quad \Longrightarrow \quad \mathsf{E}_{\mathbf{y}}\left(e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}\middle|\mathcal{F}_{\sigma_{k-1}}\right) < +\infty. \tag{3.28}$$

From now on, we assume $\lambda < \frac{9}{8}a^2$ and $\delta < \frac{2}{3a}$.

3.6. Lower bound for the exponential moment of $\tau_1 \wedge \sigma_k - \sigma_{k-1}$ for even k.

We replace in the definition of $\mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1}{\lambda}\middle|\mathcal{F}_{\sigma_{k-1}}\right)$ the stopping time τ_1, which is expressed in the terms of the minimum of $|Y_1 - Y_2|^2$ and $|Y_1 - Y_3|^2$, by another one, expressed in the terms of U_1.

Note that if $\tau_1 \in [\sigma_{k-1}, \sigma_k)$ with k even, then $|U_{23}(\tau_1)|^2 \le \frac{1}{2} + \delta$ and thanks to (3.15)

$$|U_1(\tau_1)|^2 \le \frac{4}{3}(1+R') + \frac{2}{3}(\frac{1}{2}+\delta) = \frac{5+4R'+2\delta}{3}.$$

Thus $\tau_1 \wedge \sigma_k \ge \rho_k \wedge \sigma_k$ for $\rho_k = \inf\{t \ge \sigma_{k-1}; |U_1(t)|^2 \le \frac{5+4R'+2\delta}{3}\}$. Observe that, because we have assumed that $\tau_1 > \sigma_{k-1}$, equality (3.15) also implies

$$|U_1(\sigma_{k-1})|^2 \ge \frac{2}{3}(1+R) - \frac{1}{3}(\frac{1}{2}+\delta') = \frac{1}{3}\,(\frac{3}{2}+2R-\delta').$$

Using the fact that $(e^{\lambda s}-1)/\lambda \ge s$ for $s, \lambda > 0$ we have

$$\begin{aligned} \mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1}{\lambda}\Big|\mathcal{F}_{\sigma_{k-1}}\right) \quad &\ge \quad \mathsf{E}_{\mathbf{y}}\left(\tau_1\wedge\sigma_k-\sigma_{k-1}\Big|\mathcal{F}_{\sigma_{k-1}}\right) \\ &\ge \quad \inf \mathsf{E}\Big(\rho\wedge\sigma\Big|U_1(0)=u_1, |U_{23}|^2(0)=\frac{1}{2}+\delta'\Big) \qquad (3.29) \end{aligned}$$

where the infimum is taken among all initial conditions u_1 such that $|u_1|^2 \ge \frac{1}{3}(\frac{3}{2}+2R-\delta')$,

$$\rho = \inf\{t \ge 0 : |U_1(t)|^2 \le \frac{5+4R'+2\delta}{3}\}$$

and

$$\sigma = \inf\{t \ge 0 : |U_{23}(t)|^2 \ge \frac{1}{2}+\delta\}.$$

Because U_1 and U_{23} are independent up to time τ_1, we can estimate the right hand side in (3.29) in the following way: for an arbitrary $Q > 0$ which will be chosen later,

$$\begin{aligned} &\inf_{|u_1|^2 \ge \frac{1}{3}(\frac{3}{2}+2R-\delta')} \mathsf{E}\Big[\rho\wedge\sigma\Big|U_1(0)=u_1, |U_{23}|^2(0)=\frac{1}{2}+\delta'\Big] \\ &\qquad \ge \mathsf{E}(\sigma\wedge Q\big||U_{23}|^2(0)=\frac{1}{2}+\delta') \inf_{|u_1|^2 \ge \frac{1}{3}(\frac{3}{2}+2R-\delta')} \mathsf{P}(\rho > Q\big|U_1(0)=u_1). \qquad (3.30) \end{aligned}$$

Let us compute a lower bound for each factor.

3.6.1. Lower bound for $\mathsf{P}(\rho > Q)$

By Itô formula $d|U_1(t)|^2 = 2(U_1(t), dB_1(t)) - 6a|U_1(t)|^2\,dt + d\,dt$,
and $d\log(|U_1(t)|^2) = 2|U_1(t)|^{-2}(U_1(t), dB_1(t)) - 6a\,dt + (d-2)|U_1(t)|^{-2}\,dt$.
Denote

$$M_t = 2\int_0^t |U_1(s)|^{-2}(U_1(s), dB_1(s)),$$

then, for $d \ge 2$,

$$\log(|U_1(t)|^2) \ge \log(|U_1(0)|^2) + M_t - 6at.$$

Note that $|U_1(s)|^2 \ge \frac{5+4R'+2\delta}{3}$ up to time ρ thus

$$\mathsf{E}\left(M_{t\wedge\rho}^2\right) = 4\mathsf{E}\int_0^{t\wedge\rho} |U_1(s)|^{-2}\,ds \le \frac{12\ t}{5+4R'+2\delta},$$

so that, by the Doob inequality,

$$\mathsf{P}\left(\sup_{s\le Q}|M_{s\wedge\rho}|\ge\sqrt{\frac{24\ Q}{5+4R'+2\delta}}\right)\le\frac{5+4R'+2\delta}{24\ Q}\mathsf{E}\left(M^2_{Q\wedge\rho}\right)\le\frac{1}{2}.$$

Then, with probability at least 1/2,

$$\inf_{s\le Q}\log(|U_1(s\wedge\rho)|^2)\ge\log(|U_1(0)|^2)-\sqrt{\frac{24\ Q}{5+4R'+2\delta}}-6aQ\ge\log(|U_1(0)|^2)-\sqrt{6Q/R'}-6aQ.$$

This means that

$$\mathsf{P}(\rho>Q|U_1(0)=u_1)\ge 1/2,\quad \forall|u_1|^2\ge\frac{1}{3}(\frac{3}{2}+2R-\delta'). \tag{3.31}$$

holds true as soon as (large) R, Q and R' are chosen in such a way that

$$\log(\frac{1}{3}(\frac{3}{2}+2R-\delta'))-\sqrt{6Q/R'}-6aQ>\log(\frac{5+4R'+2\delta}{3}).$$

i.e.

$$\frac{3}{2}+2R-\delta'>e^{\sqrt{6Q/R'}+6aQ}(5+4R'+2\delta). \tag{3.32}$$

3.6.2. Lower bound for $\mathsf{E}(\sigma\wedge Q)$.

We have $\sigma\wedge Q=\sigma-(\sigma-Q)1\!\!1_{\sigma>Q}\ge\sigma-(\sigma-Q)\frac{\sigma}{Q}$ hence

$$\mathsf{E}(\sigma\wedge Q)\ge 2\mathsf{E}(\sigma)-\frac{1}{Q}\mathsf{E}(\sigma^2)\,.$$

The comparison argument developped in section 3.5 leads to

$$\mathsf{E}(\sigma\wedge Q)\ge\frac{2}{2+4\delta}\mathsf{E}(\hat\sigma_2)-\frac{1}{Q}\mathsf{E}(\hat\sigma_1^2). \tag{3.33}$$

We need a lower bound for the first moment of $\hat\sigma_2$ and an upper bound for the second moment of $\hat\sigma_1$. To this end, we use the exponential moment of $\hat\sigma$ given by (3.27) for $C\neq 0$ with $-C\delta<1$, on a neibourhood of zero for λ. Differentiating twice in (3.27) at $\lambda=0$, we obtain the first and second moment of $\hat\sigma$. In order to simplify the derivative computations, from now on we make the simplifying choice

$$\delta'=\delta/2\,.$$

We obtain

$$\mathsf{E}_{\frac12+\delta'}(\hat\sigma)=-\frac{\Psi'(1)}{C^2}=\frac{e^{-2C\delta}-e^{-C\delta}}{2C^2}+\frac{\delta}{2C}=\frac{3}{4}\delta^2-\frac{7}{12}C\delta^3+\frac{1}{2C^2}\sum_{k=4}^{+\infty}\frac{(-C\delta)^k}{k!}(2^k-1)$$

The r.h.s. series is positive as soon as $C\delta\le 2$ because the sequence $u_k:=\frac{2^k-1}{k!}$ satisfies $u_k<2u_{k+1}$ for all $k\ge 4$. So, since $C_2=\frac{d}{2}$, if $\delta\le\frac{4}{d}$

$$\mathsf{E}_{\frac12+\delta'}(\hat\sigma_2)\ge\frac{3}{4}\delta^2-\frac{7}{12}C_2\delta^3=\frac{\delta^2}{12}(9-7\frac{d}{2}\delta)$$

and in particular

$$\mathsf{E}_{\frac{1}{2}+\delta'}\left(\hat{\sigma}_2\right) \geq \frac{\delta^2}{6} \quad \text{as soon as} \quad \delta \leq \frac{2}{d}$$

Moreover

$$\mathsf{E}_{\frac{1}{2}+\delta'}\left(\hat{\sigma}^2\right) = \frac{\Psi''(1) - \Psi'(1)}{C^4} = \frac{-1}{C^3}\int_{\delta'}^{\delta}(e^{-2Cx}-1)(e^{-2C\delta}+2C\delta+1) + 2Cx(e^{-2Cx}+1)dx.$$

For any negative C_1 such that $-C_1\delta < 1$ one has

$$\mathsf{E}_{\frac{1}{2}+\delta'}\left(\hat{\sigma}_1^2\right) \leq \frac{-1}{C_1^3}\int_{\delta'}^{\delta}(e^{-2C_1x}-1)(e^{-2C_1\delta}+1)dx \leq \frac{\delta}{2(-C_1)^3}(e^{-4C_1\delta}-1) \leq 2\frac{\delta^2}{C_1^2}e^4.$$

because $e^{4x} - 1 \leq 4e^4x$ for x between 0 and 1.
Since $C_1 = \frac{-3a}{2}$, inequality (3.33) leads to

$$\mathsf{E}_{\frac{1}{2}+\delta'}(\sigma \wedge Q) \geq \frac{\delta^2}{6(1+2\delta)} - \frac{8\delta^2}{9a^2Q}e^4. \tag{3.34}$$

under the conditions that $\delta \leq \min(\frac{2}{3a}, \frac{2}{d})$.
Using (3.29), (3.30), (3.31) and (3.34), we have obtained

$$\mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1}{\lambda}\Big|\mathcal{F}_{\sigma_{k-1}}\right) \geq \frac{\delta^2}{2}\left(\frac{1}{6+12\delta} - \frac{8e^4}{9a^2Q}\right) \tag{3.35}$$

This induces our choice of Q to simplify the r.h.s. of (3.35):

$$Q = \frac{32(1+2\delta)e^4}{3a^2} \quad \Leftrightarrow \quad \frac{8e^4}{9a^2Q} = \frac{1}{12+24\delta}.$$

and we obtain the lower bound

$$\mathsf{E}_{\mathbf{y}}\left(\frac{e^{\lambda(\tau_1\wedge\sigma_k-\sigma_{k-1})}-1}{\lambda}\Big|\mathcal{F}_{\sigma_{k-1}}\right) \geq \frac{\delta^2}{24(1+2\delta)} \quad \text{if} \quad \delta < \frac{2}{3a} \quad \text{and} \quad \delta \leq \frac{2}{d}. \tag{3.36}$$

3.7. Choice of the parameters

Recall that $\delta' = \delta/2$. We have to choose four parameters δ, R, R' and λ, which should satisfy the following five conditions :

$$\delta < \frac{2}{3a} \quad \text{and} \quad \delta \leq \frac{2}{d} \qquad \text{from (3.36)}$$

$$\lambda < \frac{9}{8}a^2 \qquad \text{from (3.28)}$$

$$\lambda \leq 6a - \frac{6d}{2R+3+\delta} \qquad \text{from (3.20)}$$

$$(R'(\lambda-6a) + 2\lambda(1+\delta) - 3a(1-2\delta) + 3d)\,\frac{\delta^2}{24(1+2\delta)} + 3\delta \leq 0$$

$$\text{i.e.}\quad R'(6a-\lambda)-2\lambda(1+\delta)+3a(1-2\delta)-3d \geq 72(\frac{1}{\delta}+2) \qquad \text{from (3.25)}$$

$$\frac{3-\delta}{2}+2R > e^{\sqrt{6Q/R'}+6aQ}(5+4R'+2\delta) \quad \text{for} \quad Q=\frac{32(1+2\delta)e^4}{3a^2} \qquad \text{from (3.32)}$$

We choose $\delta = \frac{2}{3a+d} \leq 1$ which complies with (3.36). We fix $\lambda = \min(a, a^2)$. Condition (3.25) is satisfied as soon as $R'(6a-\lambda) \geq 72(\frac{3a+d}{2}+2)+3d-3a+6a\delta+2\lambda+\frac{4\lambda}{3a+d}$. Since $6a\delta \leq 4$ and $\lambda \leq a$, (3.25) holds in particular if

$$R' = \frac{22a+8d+30}{a}$$

The last parameter R will be taken large enough to satisfy $2R \geq e^{\sqrt{6Q/R'}+6aQ}(7+4R')$ which implies (3.32). First remark that $R' > 22$ with our choice, hence

$$\sqrt{6Q/R'}+6aQ \leq \sqrt{\frac{64(1+2\delta)e^4}{22a^2}}+\frac{64(1+2\delta)e^4}{a} \leq \frac{10505}{a}$$

Noticing that $7+4R' \leq (95a+32d+120)/a$ with the choice of R' we made, we obtain a sufficient condition for (3.32) to hold:

$$R \geq \frac{48a+16d+60}{a}e^{10505/a}.$$

Such an R satisfies $R > 16d/a$ hence is more than sufficient for (3.20) to hold.

This completes the proofs of Lemma 3.22 and Proposition 3.17, hence Proposition 3.11 holds. This in turn completes the proof of Proposition 3.4.

Appendix A: D has a Lipschitz boundary

The following lemma is useful to apply results from R. F. Bass and P. Hsu (1990); Z. Q. Chen, P. J. Fitzsimmons and R. J. Williams (1993); M. Fukushima and M. Tomisaki (1996) to the hard ball process X.

Lemma A.1. *The domain*

$$D = \{\mathbf{x} \in (\mathbb{R}^d)^n \,;\, |x_i - x_j| > r \ \textit{for all}\ i \neq j\}\,.$$

has a Lipschitz boundary.

Proof. Define the function f_{ij} on $(\mathbb{R}^d)^n$ by $f_{ij}(\mathbf{x}) = |x_i-x_j|^2 - r^2$. Fix $\mathbf{x} \in \partial D$. Proceeding like in the proof of Proposition 4.1 in M. Fradon (2010), one can show that there exits a unit vector $\mathbf{v}$ such that, for each pair (i,j) of colliding balls of $\mathbf{x}$, $\nabla f_{ij}(\mathbf{x}).\mathbf{v} \geq \frac{r}{n\sqrt{2n}} > 0$. Indeed $\mathbf{v}$ is the direction in which each colliding ball goes away from the gravity center of the collision.

Take $m := n\ d$. By continuity,

$$\varepsilon(\mathbf{x}) = \inf\{|\mathbf{x}'-\mathbf{x}|\ ,\ \mathbf{x}' \in \partial D \text{ and } \exists (i,j) \text{ s.t. } |x_i-x_j| > r \text{ and } |x'_i - x'_j| = r\}$$

is positive. On the ball with center $\mathbf{x}$ and radius $\varepsilon(\mathbf{x})$, we choose an orthonormal coordinate system $(y_1,\ldots,y_m)$ with point $\mathbf{x}$ as the origin and direction $\mathbf{v}$ as the last axis, i.e. $\mathbf{x}'$ has coordinates $(y_1,\ldots,y_m)$ with $y_m = (\mathbf{x}'-\mathbf{x}).\mathbf{v}$.

Let us write $f_{ij} \circ h$ for the function f_{ij} expressed in this coordinates system. The partial derivative of f_{ij} at the origin with respect to the last coordinate is given by

$$\frac{\partial(f_{ij} \circ h)}{\partial y_m}(0) = \lim_{\eta \to 0} \frac{f_{ij}(\mathbf{x} + \eta \mathbf{v}) - f_{ij}(\mathbf{x})}{\eta} = \nabla f_{ij}(\mathbf{x}).\mathbf{v} > 0.$$

Therefore, due to the implicit function theorem, there exists a $\mathcal{C}^1$-function g_{ij} on $\mathbb{R}^m$ such that $f_{ij}(h(y_1, \ldots, g_{ij}(y_1, \ldots, y_{m-1}, \cdot))) = Id$ for each $h(y_1, \ldots, y_m) \in B(\mathbf{x}, \varepsilon)$ where $\varepsilon < \varepsilon(\mathbf{x})$ is such that, on $B(\mathbf{x}, \varepsilon)$, all the functions $\nabla f_{ij}(\cdot).\mathbf{v}$ stay positive for any pair (i,j) of colliding balls of $\mathbf{x}$. The maps $y_m \mapsto f_{ij}(h(y_1, \ldots, y_m))$ are increasing, so that for $h(y_1, \ldots, y_m)$ in $B(\mathbf{x}, \varepsilon)$:

$$\begin{aligned}
& y_m > \max\{g_{ij}(y_1, \ldots, y_{m-1}, 0) \text{ s.t. } |x_i - x_j| = r\} \\
\Leftrightarrow \quad & \forall i < j \text{ s.t. } |x_i - x_j| = r,\ y_m > g_{ij}(y_1, \ldots, y_{m-1}, 0) \\
\Leftrightarrow \quad & \forall i < j, \text{ s.t. } |x_i - x_j| = r,\ f_{ij}(h(y_1, \ldots, y_{m-1}, y_m)) > f_{ij}(h(y_1, \ldots, g_{ij}(y_1, \ldots, y_{m-1}, 0))) = 0 \\
\Leftrightarrow \quad & \forall i < j \quad f_{ij}(h(y_1, \ldots, y_{m-1}, y_m)) > 0 \\
\Leftrightarrow \quad & h(y_1, \ldots, y_m) \in D
\end{aligned}$$

Note that, since the g_{ij} are $\mathcal{C}^1$, they are Lipschitz continuous with Lipschitz constant C_{ij}, which leads to the Lipschitz continuity of the function

$$(y_1, \ldots, y_{m-1}) \mapsto\longmapsto \max\{g_{ij}(y_1, \ldots, y_{m-1}, 0) \text{ for } (i,j) \text{ with } |x_i - x_j| = r\}.$$

Indeed

$$\begin{aligned}
\max_{\{i<j:\ |x_i - x_j| = r\}} g_{ij}(y) - \max_{\{i<j:\ |x_i - x_j| = r\}} g_{ij}(y') &= g_{i_0 j_0}(y) - \max_{\{i<j:\ |x_i - x_j| = r\}} g_{ij}(y') \text{ for some } i_0, j_0 \\
&\leq g_{i_0 j_0}(y) - g_{i_0 j_0}(y') \leq C_{i_0 j_0}|y - y'| \\
&\leq \big(\max_{\{i<j:\ |x_i - x_j| = r\}} C_{ij}\big)|y - y'|
\end{aligned}$$

Hence D is a Lipschitz domain. □

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